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KHINCHIN TYPE CONDITION FOR TRANSLATION SURFACES AND ASYMPTOTIC LAWS FOR THE TEICHMÜLLER FLOW

BY LUCA MARCHESE

ABSTRACT. — We study a diophantine property for translation surfaces, defined in terms of saddle connections and inspired by classical Khinchin condition. We prove that the same dichotomy holds as in Khinchin theorem, then we deduce a sharp estimate on how fast the typical Teichmüller geodesic wanders towards infinity in the moduli space of translation surfaces. Finally we prove some stronger result in genus one.

RÉSUMÉ (*Conditions de type Khinchin pour les surfaces de translation et lois asymptotiques pour le flot de Teichmüller*)

On étude une propriété diophantienne pour les surfaces de translation, définie en termes de connexions de selles et inspirée par la condition de Khinchin classique. On prouve la même dichotomie du théorème de Khinchin et on en déduit une estimation sur la vitesse des excursions à l'infini pour une géodésique de Teichmüller typique dans l'espace des modules des surfaces de translation. Enfin on preuve un résultat plus fort en genre un.

1. Introduction

The moduli space of flat tori is identified with the *modular surface*, that is the quotient of the hyperbolic half plane $\mathbb{H}$ by the action of $\mathrm{PSL}(2,\mathbb{Z})$ via *Moebius transformations*, which is homeomorphic to a punctured sphere and has finite

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area. A neighborhood of the cusp corresponds to those flat tori with a very short closed geodesic, or equivalently to points in the standard fundamental domain with big imaginary part. The geodesic flow g_t acts ergodically on the unitary tangent bundle of $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$, therefore a generic geodesic makes infinitely many excursions to the cusp. The asymptotic rate of this phenomenon is measured by the so-called *logarithmic law*.

THEOREM. — *For any point z in $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ and almost any unitary tangent vector v at z, if g_t is the geodesic at z in the direction of v and d denotes the* Poincaré distance, *we have*

$$\limsup_{t\to\infty} \frac{d(g_t(z),z)}{\log t} = 1/2.$$

The logarithmic law has been generalized in many settings, in particular Sullivan proved it for the geodesic flow on manifolds with negative curvature(see [13])and Masur proved a logarithmic law for *Teichmüller geodesics* on the moduli space of complex curves of any genus(see [11]).

The geodesic flow on $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ has a well know relation with the *continued fraction algorithm*, which have been described in [12], and in general with arithmetics. In particular the logarithmic law is strictly related to Khinchin Theorem(see [5]), which concerns the general diophantine condition on a real number α defined in terms of the set of integers n such that:

$$\{n\alpha\} < \varphi(n),$$

where $\{\cdot\}$ denotes the fractionary part and $\varphi : \mathbb{N} \to \mathbb{R}_+$ is a positive (vanishing) sequence.

THEOREM (Khinchin). — *Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ be a positive sequence.*

- *If $\sum_{n=1}^{+\infty} \varphi(n) < +\infty$ then for almost any α there are just finitely many $n \in \mathbb{N}$ such that $\{n\alpha\} < \varphi(n)$.*
- *If $n\varphi(n)$ is monotone decreasing and $\sum_{n=1}^{+\infty} \varphi(n) = +\infty$ then for almost any α there are infinitely many $n \in \mathbb{N}$ such that $\{n\alpha\} < \varphi(n)$.*

Translation surfaces. — The natural generalization to higher genus of a flat torus brings to the notion of *translation surface*, that is a compact, orientable and boundary-less flat surface X, with conical singularities whose angle is a multiple of 2π. If g is the genus of the surface, $\Sigma = \{p_1, \dots, p_r\}$ is the set of conical singularities and $k_1, \dots, k_r$ are positive integers such that for any $i = 1, \dots, r$ the angle at p_i is $2k_i\pi$, then we have the relation $k_1 + \dots + k_r = 2g + r - 2$. Flat neighborhoods in X are naturally identified with open sets in $\mathbb{C}$, that is they admit a local coordinate z, thus $X \setminus \Sigma$ inherits the structure of Riemann surface and it is easy to see that the structure extends to X. Since the angles at conical singularities are multiples of 2π then dz defines a complex one form on $X \setminus \Sigma$, which extends by $z^{k_i-1}dz$ at any point $p_i \in \Sigma$, that is it has a zero of order $k_i - 1$. The datum of a translation surface is therefore equivalent to the datum of a Riemann surface together with a complex one form. The *moduli space* of translation surfaces of fixed genus g admits a *stratification*, where a *stratum* is the set $\mathscr{H}(k_1, \ldots, k_r)$ of those translation surfaces with r conical singularities with fixed values $2k_1\pi, \ldots, 2k_r\pi$ for the angles. We assume that the singularities are labeled. Any $\mathscr{H}(k_1, \ldots, k_r)$ is a complex orbifold with complex dimension $2g + r - 1$, that is it is locally homeomorphic to the quotient of $\mathbb{C}^{2g+r-1}$ by the action of a finite group, orbifold points occurring at those translation surfaces whose underling complex structure admits non-trivial automorphisms. In general strata are neither compact nor connected and their connected components have been classified in [6].

A *saddle connection* for a translation surface X is a geodesic $\gamma : (0, T) \to X$ for the flat metric such that $\gamma^{-1}(\Sigma) = \{0, T\}$, that is γ starts and ends in Σ and it does not contain any other conical singularity in its interior. If γ is a saddle connection for X we define a complex number $\mathrm{Hol}(\gamma) := \int_\gamma w_X$, which is called the *holonomy* of γ, where w_X is the holomorphic one form associated to X. We call $\mathrm{Hol}(X)$ the set of complex numbers $v = \mathrm{Hol}(\gamma)$, where γ varies among the saddle connections of X. It is possible to see that $\mathrm{Hol}(X)$ contains pure imaginary elements only for translation surfaces X lying in a codimension one subset of $\mathscr{H}(k_1, \ldots, k_r)$, thus in particular a zero measure subset. Nevertheless $\mathrm{Hol}(X)$ is always dense in $\mathbb{PR}^2$ and in particular it accumulates to the imaginary axis. We define a diophantine condition comparing the deviation from the imaginary axis of elements of $\mathrm{Hol}(X)$ with their norm. For a monotone decreasing function $\varphi : (0, +\infty) \to (0, +\infty)$ and for an element v in $\mathrm{Hol}(X)$ we consider the condition

$$|\Re(v)| < \varphi(|v|). \tag{1.1}$$

For any translation surface X the associated one form w_X induces a pair of parallel vector fields ∂_x and ∂_y on $X \setminus \Sigma$ defined by $w_X(\partial_x) = 1$ and $w_X(\partial_y) = \sqrt{-1}$ and called respectively *horizontal* and *vertical* vector field. They are not complete, since their trajectories stop if they arrive at a point of Σ. In particular any point $p_i \in \Sigma$ is the starting point of exactly k_i half-trajectories of ∂_x, which are called *horizontal separatrices*. A *frame* for a translation surface X is the datum of r different horizontal separatrices $(S_1, \ldots, S_r)$, such that S_i starts at p_i for any $i \in \{1, \ldots, r\}$. Any X admits $\prod_{i=1}^r k_i$ different choices of a frame. We denote $\widehat{X}$ the datum $(X, S_1, \ldots, S_r)$ of a translation surface with frame and we call $\widehat{\mathscr{H}}(k_1, \ldots, k_r)$ the stratum of the moduli space of translation surfaces with frame, which is a covering space of $\mathscr{H}(k_1, \ldots, k_r)$ with natural projection $\widehat{X} \mapsto X$. It is possible to show that the covering is non-trivial, that is it does not admit a continuous inverse $\mathscr{H}(k_1, \ldots, k_r) \to \widehat{\mathscr{H}}(k_1, \ldots, k_r)$, thus a continuous choice of a frame is not possible on $\mathcal{H}(k_1,\dots,k_r)$ but only on $\widehat{\mathcal{H}}(k_1,\dots,k_r)$. Conceptually the construction is the same as that of the orientable double covering of a non-orientable manifold. Note that the space $\widehat{\mathcal{H}}(k_1,\dots,k_r)$ has also been considered by C. Boissy in [1], where the author computes the number of connected components of strata of translation surfaces with frame.

Let $\widehat{X}$ be a translation surface with frame whose frame is $(S_1,\dots,S_r)$. Let p_j and p_i be any two points in Σ(possibly the same)and let (m,l) be a pair of integers with $1\leq m\leq k_j$ and $1\leq l\leq k_i$. We define the *bundle* $\mathcal{C}^{(p_j,p_i,m,l)}(\widehat{X})$ as the set of those saddle connections γ for X which start at p_j, end in p_i and satisfy

$$2(m-1)\pi\leq \mathrm{angle}(\gamma,S_j)<2m\pi \texttt{ and } 2(l-1)\pi\leq \mathrm{angle}(\gamma,S_i)<2l\pi,$$

where S_j and S_i are the horizontal separatrices in the frame starting respectively at p_j and p_i. A choice of a frame for X therefore induces a decomposition of $\mathrm{Hol}(X)$ into subsets

$$\mathrm{Hol}^{(p_j,p_i,m,l)}(\widehat{X}):=\{\mathrm{Hol}(\gamma);\gamma\in\mathcal{C}^{(p_j,p_i,m,l)}(\widehat{X})\}.$$

We prove the following dichotomy.

THEOREM 1.1. — *Let $\varphi:(0,+\infty)\to(0,+\infty)$ be a positive function.*

1. *If $\varphi(t)$ is decreasing monotone and $\int_1^{+\infty}\varphi(t)dt<+\infty$ then* $\mathrm{Hol}(X)$ *contains just finitely many solutions of Equation* (1.1) *for almost any X in $\mathcal{H}(k_1,\dots,k_r)$.*
2. *If $t\varphi(t)$ is monotone decreasing and $\int_1^{+\infty}\varphi(t)dt=+\infty$ then for almost any $\widehat{X}$ in $\widehat{\mathcal{H}}(k_1,\dots,k_r)$, for any pair of points p_j,p_i in Σ and for any pair of integers (m,l) with $1\leq m\leq k_j$ and $1\leq l\leq k_i$ the set* $\mathrm{Hol}^{(p_j,p_i,m,l)}(\widehat{X})$ *contains infinitely many solutions of Equation* (1.1).

We note that the function φ in Theorem 1.1 is defined on $(0,+\infty)$, anyway we consider its integral just over $(1,+\infty)$ because we are just interested by convergence or divergence at infinity.

Translation surfaces are strictly linked to *interval exchange transformations*(i.e.t.s in the following), a class of maps of the interval which has been largely studied, for example in [14], [9], [17]. In particular the second part of Theorem 1.1 (divergent integral) is a consequence of a generalization of Khinchin Theorem to i.e.t.s which is proved in [7]. On the other hand the first part of Theorem 1.1 (convergent integral) admits a stronger statement, which is proved independently from i.e.t.s with an easier argument. Such argument is based on a natural construction, which consists in fixing a translation surface X and rotating its vertical direction. If θ is the rotation angle, we call X_θ the rotated translation surface. The image of the application $\theta\mapsto X_\theta$ is an