ISSN 0012-9593

quatrième série - tome 41 *fascicule 3* *mai-juin 2008*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Laura G. DeMARCO & Curtis T. McMULLEN

Trees and the dynamics of polynomials

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 41, 2008, p. 337 à 383

TREES AND THE DYNAMICS OF POLYNOMIALS*

BY LAURA G. DEMARCO AND CURTIS T. MCMULLEN

ABSTRACT. – In this paper we study branched coverings of metrized, simplicial trees $F : T \to T$ which arise from polynomial maps $f : \mathbb{C} \to \mathbb{C}$ with disconnected Julia sets. We show that the collection of all such trees, up to scale, forms a contractible space $\mathbb{P}T_D$ compactifying the moduli space of polynomials of degree D; that F records the asymptotic behavior of the multipliers of f; and that any meromorphic family of polynomials over Δ^* can be completed by a unique tree at its central fiber. In the cubic case we give a combinatorial enumeration of the trees that arise, and show that $\mathbb{P}T_3$ is itself a tree.

RÉSUMÉ. – Dans ce travail, nous étudions des revêtements ramifiés d'arbres métriques simpliciaux $F : T \to T$ qui sont obtenus à partir d'applications polynomiales $f : \mathbb{C} \to \mathbb{C}$ possédant un ensemble de Julia non connexe. Nous montrons que la collection de tous ces arbres, à un facteur d'échelle près, forme un espace contractile $\mathbb{P}T_D$ qui compactifie l'espace des modules des polynômes de degré D. Nous montrons aussi que F enregistre le comportement asymptotique des multiplicateurs de f et que toute famille méromorphe de polynômes définis sur Δ^* peut être complétée par un unique arbre comme sa fibre centrale. Dans le cas cubique, nous donnons une énumération combinatoire des arbres ainsi obtenus et montrons que $\mathbb{P}T_3$ est lui-même un arbre.

1. Introduction

The basin of infinity of a polynomial map $f : \mathbb{C} \to \mathbb{C}$ carries a natural foliation and a flat metric with singularities, determined by the escape rate of orbits. As f diverges in the moduli space of polynomials, this Riemann surface collapses along its foliation to yield a metrized simplicial tree (T, d), with limiting dynamics $F : T \to T$.

In this paper we characterize the trees that arise as limits, and show they provide a natural boundary $\mathbb{P}\mathcal{T}_D$ compactifying the moduli space of polynomials of degree D. We show that

*Research of both authors supported in part by the NSF..

(T, d, F) records the limiting behavior of the multipliers of f at its periodic points, and that any degenerating analytic family of polynomials $\{f_t(z) : t \in \Delta^*\}$ can be completed by a unique tree at its central fiber. Finally we show that in the cubic case, the boundary of moduli space $\mathbb{P}\mathcal{T}_3$ is itself a tree; and for any D, $\mathbb{P}\mathcal{T}_D$ is contractible.

The metrized trees (T, d, F) provide a counterpart, in the setting of iterated rational maps, to the $\mathbb{R}$-trees that arise as limits of hyperbolic manifolds.

The quotient tree. – Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $D \geq 2$. The points $z \in \mathbb{C}$ with bounded orbits under f form the compact *filled Julia set*

$$K(f) = \{z : \sup_n |f^n(z)| < \infty\};$$

its complement, $\Omega(f) = \mathbb{C} \setminus K(f)$, is the *basin of infinity*. The *escape rate* $G : \mathbb{C} \to [0, \infty)$ is defined by

$$G(z) = \lim_{n \to \infty} D^{-n} \log^+ |f^n(z)|;$$

it is the Green function for $K(f)$ with a logarithmic pole at infinity. The escape rate satisfies $G(f(z)) = DG(z)$, and thus it gives a semiconjugacy from f to the simple dynamical system $t \mapsto Dt$ on $[0, \infty)$.

Now suppose that the Julia set $J(f) = \partial K(f)$ is disconnected; equivalently, suppose that at least one critical point of f lies in the basin $\Omega(f)$. Then some fibers of G are also disconnected, although for each $t > 0$ the fiber $G^{-1}(t)$ has only finitely many components.

To record the combinatorial information of the dynamics of f on $\Omega(f)$, we form the *quotient tree* $\overline{T}$ by identifying points of $\mathbb{C}$ that lie in the same connected component of a level set of G. The resulting space carries an induced dynamical system

$$F : \overline{T} \to \overline{T}.$$

The escape rate G descends to give the *height function* H on $\overline{T}$, yielding a commutative diagram

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi"'] \arrow[dr, "G"] & \\ \overline{T} \arrow[r, "H"'] & {[0,\infty)} \end{tikzcd}$$

respecting the dynamics. See Figure 1 for an illustration of the trees in two examples. Note that only a finite subtree of $\overline{T}$ has been drawn in each case.

The *Julia set* of $F : \overline{T} \to \overline{T}$ is defined by

$$J(F) = \pi(J(f)) = H^{-1}(0).$$

It is a Cantor set with one point for each connected component of $J(f)$. With respect to the measure of maximal entropy μ_f, the quotient map $\pi : J(f) \to J(F)$ is almost injective, in the sense that μ_f-almost every component of $J(f)$ is a single point (Theorem 3.2). In particular, there is no loss of information when passing to the quotient dynamical system:

Theorem 1.1. – *Let f be a polynomial of degree $D \geq 2$ with disconnected Julia set. The measure-theoretic entropy of $(f, J(f), \mu_f)$ and its quotient $(F, J(F), \pi_*(\mu_f))$ are the same — they are both* $\log D$.

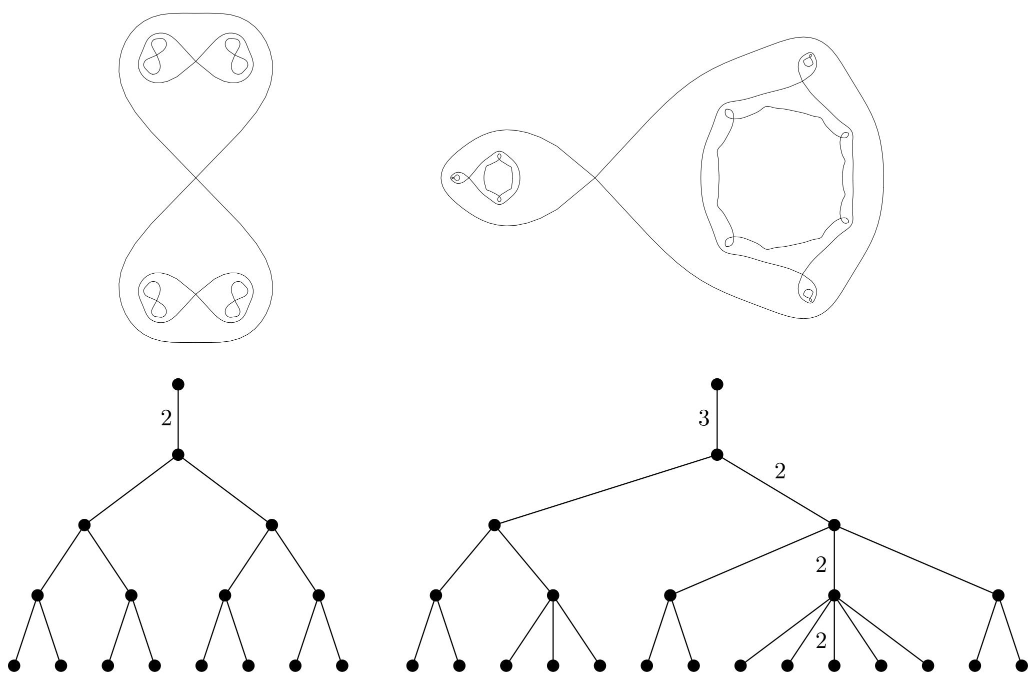

FIGURE 1. Critical level sets of $G(z)$ for $f(z) = z^2 + 2$ and $f(z) = z^3 + 2.3z^2$, and the corresponding combinatorial trees. The edges mapping with degree > 1 are indicated. The Julia set of $z^2 + 2$ is a Cantor set, while the Julia set of $z^3 + 2.3z^2$ contains countably many Jordan curves.

The quotient of the basin of infinity,

$$T = \pi(\Omega(f)) = H^{-1}(0, \infty),$$

is an open subset of $\overline{T}$ homeomorphic to a simplicial tree. In fact T carries a canonical simplicial structure, determined by the conditions:

1. $F : T \to T$ is a simplicial map,
2. the vertices of T consist of the grand orbits of the branch points of T, and
3. the height function H is linear on each edge of T.

Details can be found in § 2.

The *height metric* d is a path metric on T, defined so that adjacent vertices of T satisfy

$$d(v, v') = |H(v) - H(v')|.$$

We refer to the triple

$$\tau(f) = (T, d, F)$$

as the *metrized polynomial-like tree* obtained as the quotient of f. The space $\overline{T}$ is the metric completion of (T, d).

In § 2, we introduce an *abstract* metrized polynomial-like tree with dynamics $F : T \to T$, and in § 7 we show:

THEOREM 1.2. – *Every metrized polynomial-like tree* (T, d, F) *arises as the quotient* $\tau(f)$ *of a polynomial* f.

Special cases of Theorem 1.2 were proved by Emerson; Theorems 9.4 and 10.1 of [7] show that any tree with just one escaping critical point (though possibly of high multiplicity) and with divergent sums of moduli can be realized by a polynomial.

Spaces of trees and polynomials. – Let $\mathcal{T}_D$ denote the space of isometry classes of metrized polynomial-like trees (T, d, F) of degree D. The space $\mathcal{T}_D$ carries a natural geometric topology, defined by convergence of finite subtrees. There is a continuous action of $\mathbb{R}_+$ on $\mathcal{T}_D$ by rescaling the metric d, yielding as quotient the projective space

$$\mathbb{P}\mathcal{T}_D = \mathcal{T}_D/\mathbb{R}_+.$$

In § 5 we show:

THEOREM 1.3. – *The space* $\mathbb{P}\mathcal{T}_D$ *is compact and contractible.*

Now let MPoly_D denote the moduli space of polynomials of degree D, the space of polynomials modulo conjugation by the affine automorphisms of $\mathbb{C}$. The conjugacy class of a polynomial f will be denoted $[f]$. The space MPoly_D is a complex orbifold, finitely covered by $\mathbb{C}^{D-1}$. The *maximal escape rate*

$$M(f) = \max\{G(c) : f'(c) = 0\}$$

depends only on the conjugacy class of f; Branner and Hubbard observed in [3] that M descends to a continuous and proper map $M : \mathrm{MPoly}_D \to [0, \infty)$.

The *connectedness locus* $\mathcal{C}_D \subset \mathrm{MPoly}_D$ is the subset of polynomials with connected Julia set; it coincides with the locus $M(f) = 0$ and is therefore a compact subset of MPoly_D. We denote its complement by

$$\mathrm{MPoly}_D^* = \mathrm{MPoly}_D \setminus \mathcal{C}_D.$$

The metrized polynomial-like tree $\tau(f) = (T, d, F)$ depends only on the conjugacy class of f, so τ induces a map

$$\tau : \mathrm{MPoly}_D^* \to \mathcal{T}_D.$$

Note that the compactness of the connectedness locus $\mathcal{C}_D$ implies that every divergent sequence in MPoly_D will eventually be contained in the domain of τ.

There is a natural action of $\mathbb{R}_+$ on MPoly_D^* obtained by 'stretching' the complex structure on the basin of infinity. In § 6 and § 8 we show:

THEOREM 1.4. – *The map* $\tau : \mathrm{MPoly}_D^* \to \mathcal{T}_D$ *is continuous, proper, surjective, and equivariant with respect to the action of* $\mathbb{R}_+$ *by stretching of polynomials and by metric rescaling of trees.*

THEOREM 1.5. – *The moduli space of polynomials admits a natural compactification* $\overline{\mathrm{MPoly}}_D = \mathrm{MPoly}_D \cup \mathbb{P}\mathcal{T}_D$ *such that*

- MPoly_D *is dense in* $\overline{\mathrm{MPoly}}_D$, *and*
- *the iteration map* $[f] \mapsto [f^n]$ *extends continuously to* $\overline{\mathrm{MPoly}}_D \to \overline{\mathrm{MPoly}}_{D^n}$.