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RATIONAL INVARIANT TORI, PHASE SPACE TUNNELING, AND SPECTRA FOR NON-SELFADJOINT OPERATORS IN DIMENSION 2*

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ABSTRACT. – We study spectral asymptotics and resolvent bounds for non-selfadjoint perturbations of selfadjoint h-pseudodifferential operators in dimension 2, assuming that the classical flow of the unperturbed part is completely integrable. Spectral contributions coming from rational invariant Lagrangian tori are analyzed. Estimating the tunnel effect between strongly irrational (Diophantine) and rational tori, we obtain an accurate description of the spectrum in a suitable complex window, provided that the strength of the non-selfadjoint perturbation $\gg h$ (or sometimes $\gg h^2$) is not too large.

RÉSUMÉ. – Nous étudions des asymptotiques spectrales et des estimations de la résolvante des perturbations non-autoadjointes d'opérateurs h-pseudodifférentiels autoadjoints en dimension 2, en supposant que le flot classique de la partie non-perturbée soit complètement intégrable. Les contributions spectrales parvenant des tores invariants lagrangiens rationnels sont analysées. En estimant l'effet tunnel entre des tores diophantiens et rationnels, nous obtenons une description précise du spectre dans une région convenable du plan complexe spectral, sous l'hypothèse que la force de la perturbation non-autoadjointe $\gg h$ (ou parfois $\gg h^2$) ne soit pas trop grande.

1. Introduction

In [24], A. Melin and the second author observed that for large and stable classes of non-selfadjoint analytic (pseudo)differential operators in two dimensions, the individual eigenvalues can be determined up to arbitrarily high powers of the semiclassical parameter by a complex Bohr-Sommerfeld quantization condition. This is quite analogous to known results in dimension one in the selfadjoint case [8], [3], [7], and remarkable in the sense that corresponding results for selfadjoint operators in higher dimensions are known only in very special situations. Applications to resonances were also given in [24].

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A natural continuation of [24] was to study non-selfadjoint perturbations of selfadjoint operators in the semiclassical limit, of the form

$$P_\varepsilon(x, hD_x) = P(x, hD_x) + i\varepsilon Q(x, hD_x),\ 0 < h \ll 1, \tag{1.1}$$

with principal symbol (classical Hamiltonian)

$$p_\varepsilon(x, \xi) = p(x, \xi) + i\varepsilon q(x, \xi), \tag{1.2}$$

either on $\mathbb{R}^2$ or on a compact analytic manifold of dimension 2. Here P is selfadjoint so p is real, and we may assume to fix the ideas that q is real. Both p and q are assumed to be analytic, at least in the cases when $\varepsilon \gg h$.

In [14]–[16] we studied the case when the classical flow of p is periodic and showed that the spectrum has a lattice structure as in [24], with an eigenvalue separation of the order of h in the real ("horizontal") direction and of the order of εh in the imaginary ("vertical") direction. (In [16] we got richer phenomena near branching point levels.) The methods in [14]–[16] were based on a reduction to a one-dimensional operator. Again it should be noticed that the results obtained are more precise than what is currently known in the case of selfadjoint perturbations. Applications to resonances and the damped wave equation were given. See also [11].

As in classical works of A. Weinstein [35] and Y. Colin de Verdière [34], the trajectory averages of q play an important role in the precise formulation of the results. Under much more general assumptions they allow to estimate the width of the spectrum in the imaginary directions (see also [19], [29]). It should also be recalled that the real parts of the eigenvalues distribute according to the same Weyl law as for the unperturbed operator P (see Markus and Matsaev [22], [21]).

The next step was taken by the authors together with S. Vũ Ngọc in [17], where we studied the case when p is classically completely integrable or close to being completely integrable. In the integrable case, the energy surface $p = E_0$ foliates into invariant Lagrangian tori and possibly some more complicated sets. The classical flow on each invariant torus has a rotation number which "most of the time" is Diophantine (i.e. poorly approximated by rational numbers). On such a torus Λ (or more generally on an irrational one), the time averages

$$\langle q\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} q \circ \exp tH_p dt$$

of q along the classical trajectories of p all converge to the space average $\langle q\rangle(\Lambda)$ of q over Λ, when $T \to \infty$. When Λ is a torus with a rational rotation number, or a more general "singular" invariant set in the foliation of the energy surface $p^{-1}(E_0)$, then we need to consider the whole interval $Q_\infty(\Lambda)$ of limits of flow averages as above, and in the rational torus case we have $\langle q\rangle(\Lambda) \in Q_\infty(\Lambda)$.

In the completely integrable case, the main result of [17] says very roughly that if $F_0 \in \mathbb{R}$ is a value such that $F_0 = \langle q\rangle(\Lambda_j)$ for finitely many Diophantine tori $\Lambda_1, ..., \Lambda_{N_0}$ in $p^{-1}(E_0)$, and F_0 does not belong to $Q_\infty(\Lambda)$ for any other invariant set Λ in the energy surface, then the spectrum can be completely determined in a rectangle $[E_0 - h^\delta/C, E_0 + h^\delta/C] + i\varepsilon[F_0 - h^\delta/C, F_0 + h^\delta/C]$ modulo $\mathcal{O}(h^\infty)$, where δ is a positive exponent that can be chosen arbitrarily small, and ε may vary in any interval of the form $h^K < \varepsilon \ll 1$. Again the eigenvalues

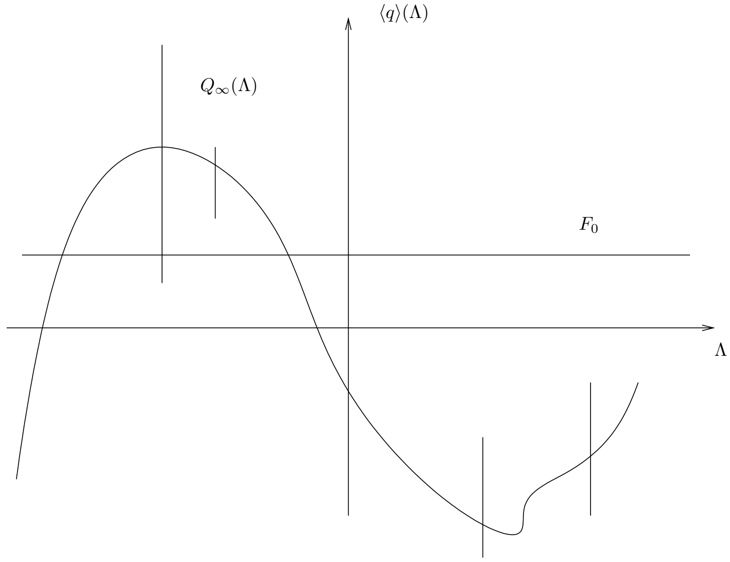

FIGURE 1. The figure represents the graph of the function $\Lambda \mapsto \langle q\rangle(\Lambda)$ as Λ varies over the set of flow invariant Lagrangian tori in the energy surface $p^{-1}(E_0)$. The vertical segments in the figure correspond to the intervals $Q_\infty(\Lambda)$ of limits of flow averages $\langle q\rangle_T$, as $T \to \infty$, when Λ is a rational invariant torus.

form a (superposition of finitely many) distorted lattice(s), with horizontal spacing $\sim h$ and vertical spacing $\sim \varepsilon h$. The proofs were based on the use of suitable exponentially weighted spaces and Birkhoff normal forms near the Diophantine tori.

Notice that the intervals $Q_\infty(\Lambda)$ shrink very fast if Λ are rational tori converging to a Diophantine torus Λ_0. For that reason, there may be plenty of levels F_0 satisfying the assumptions of the result above, and we may cover a substantial fraction of the energy band $[E_0 - 1/C, E_0 + 1/C] + i\varepsilon[\liminf\langle q\rangle_T, \limsup\langle q\rangle_T]$ with such rectangles.

Nevertheless, the intervals $Q_\infty(\Lambda)$ for the rational tori form sets of positive measure of forbidden values for the main result of [17]. In the present paper we shall study what happens when F_0 belongs to finitely many such intervals. Our first attempt was to use secular perturbation theory to analyze the individual eigenvalues produced by rational tori. However this leads to possibly quite serious pseudospectral phenomena for certain one-dimensional operators, and it is doubtful whether such a program can succeed completely. Instead we estimated the number of eigenvalues that can be created by such tori and showed that it is much smaller than the number of eigenvalues created by the Diophantine ones.

Very roughly, the main result of the present paper is as follows: assume that F_0 is equal to $\langle q\rangle(\Lambda_{j,d})$ for finitely many Diophantine tori $\Lambda_{j,d}$ (as in the main result of [17]), and that $F_0 \in Q(\Lambda_{k,r}) \setminus \langle q\rangle(\Lambda_{k,r})$ for finitely many rational tori $\Lambda_{k,r}$. We further assume that F_0 belongs to no other set $Q_\infty(\Lambda)$, for Λ in the foliation of $p^{-1}(E_0)$, and we restrict ε to the interval

$$h \ll \varepsilon \leq h^{\frac{2}{3}+\delta}, \tag{1.3}$$

where $\delta > 0$ is any fixed parameter. Then the spectrum of P_ε in the rectangle

$$R(\varepsilon) = [E_0 - \varepsilon/C, E_0 + \varepsilon/C] + i\varepsilon[F_0 - \varepsilon^\delta, F_0 + \varepsilon^\delta] \tag{1.4}$$

is of the form $E_d \cup E_r$, where

- E_d is the union of finitely many distorted lattices as in the main result of [17] (see above)
- E_r is a set of cardinality $\mathcal{O}(\varepsilon^{3/2}/h^2)$.

Here we notice that E_d is of cardinality $\sim \varepsilon^{1+\delta}/h^2$, so choosing δ small enough we see that most eigenvalues in the rectangle $R(\varepsilon)$ belong to E_d and can be asymptotically determined.

Using secular theory arguments ("partial Birkhoff normal forms") we simplify the operator near each rational torus and conclude roughly that the eigenvalues in $R(\varepsilon)$ produced near the rational tori must come from a set in phase space of volume $\mathcal{O}(\varepsilon^{3/2})$. In the absence of Diophantine tori, this leads to the bound $\mathcal{O}(\varepsilon^{3/2}h^{-2})$ on the total number of eigenvalues in $R(\varepsilon)$. When Diophantine tori are present this has to be combined with the analysis of [17], via an auxiliary so called Grushin problem. Near the Diophantine tori we have a nice control on the solution operator, while near the rational tori, we only have the bound $\mathcal{O}(\exp{(C\varepsilon^{3/2}/h^2)})$. Luckily, by means of phase space exponential weights we are able to estimate the tunnel effect between the tori by $\mathcal{O}(\exp{(-1/(Ch))})$, and thanks to the condition (1.3) the Grushin problem can be solved globally, leading to the result above.

In a parallel work [13], the first author and San Vũ Ngọc are currently investigating the case of larger real perturbations. Here the strategy is quite different and uses KAM theory to show that the rational tori split into Diophantine ones under the effect of the perturbation.

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2. Statement of the main results

2.1. General assumptions

We shall start by describing the general assumptions on our operators, which will be the same as in [17], as well as in the earlier papers mentioned above. Let M denote either the space $\mathbb{R}^2$ or a real analytic compact manifold of dimension 2. We shall let $\widetilde{M}$ stand for a complexification of M, so that $\widetilde{M} = \mathbb{C}^2$ in the Euclidean case, and in the compact case, we let $\widetilde{M}$ be a Grauert tube of M — see [6] for the definition and further references.

When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P^w(x, hD_x, \varepsilon; h), \quad 0 < h \leq 1, \tag{2.1}$$