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BILINEAR VIRIAL IDENTITIES AND APPLICATIONS

BY Fabrice PLANCHON* AND Luis VEGA**

ABSTRACT. – We prove bilinear virial identities for the nonlinear Schrödinger equation, which are extensions of the Morawetz interaction inequalities. We recover and extend known bilinear improvements to Strichartz inequalities and provide applications to various nonlinear problems, most notably on domains with boundaries.

RÉSUMÉ. – On démontre des identités de type viriel bilinéaire pour l'équation de Schrödinger non-linéaire, qui peuvent être vues comme des extensions des inégalités d'interaction de Morawetz. Ceci permet de retrouver et d'étendre des raffinements bilinéaires des inégalités de Strichartz, et nous donnons également des applications à plusieurs problèmes non-linéaires, notamment sur les domaines à bord.

1. Introduction

Dispersive estimates are known to be an essential tool in dealing with low regularity well-posedness issues for the nonlinear Schrödinger equation. Among the most useful ones are Strichartz inequalities: starting with [26], they were completed by [15] and finally by [19]. As space-time bounds for solutions to the linear Schrödinger equation in $\mathbb{R}^n$, they are closely related to the Fourier restriction problem in harmonic analysis, and as such heavily rely on the use of Fourier transform techniques. Extensions of these inequalities to more complicated geometrical settings have been the subject of intense research over the last decade, to the point where quoting all possible references would fill this page. It should be noted that these works are based on appropriate refinements of the $\mathbb{R}^n$ case, through Fourier Integral Operator, FBI, wave packet or any appropriate microlocal generalizations of Fourier analysis (for a notable exception using vector field methods, see [23]). On the other hand, one has virial type identities, of which the Morawetz identity (proved by Lin-Strauss [21]) is perhaps the most well-known: such identities have two key features, they are obtained by integration

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by parts and they usually apply to the nonlinear equation. We remark that the local smoothing effect, which came much later and was first observed in the flat case (see [14], [24], [29]), may be seen as part of this category as well, though proofs usually require a sophisticated "integration by parts" involving pseudo-differential operators or resolvent methods. A new kind of inequality was introduced in [12], the Morawetz interaction inequality, which seemed to have the benefit of both worlds: one may recover a specific, non-sharp Strichartz estimate and it also applies to the nonlinear equation (providing an essential tool to solve the H^1-critical defocusing NLS in 3D, [13]). Subsequent developments include a curved space version ([17]) and a quartic interaction inequality for NLS on $\mathbb{R}$ ([11]).

In the present work, we explore a different direction, which builds upon the understanding of the local smoothing effect and its fundamentally 1D nature. This naturally leads to a new set of identities with several interesting consequences:

- in 1D, one recovers, by a simple argument, an identity of [22], which implies the Fefferman-Stein inequality in its bilinear version; from there the (almost) full set of Strichartz/maximal function estimates may be derived. More importantly, we get a nonlinear identity.
- In 2D and higher, one obtains an $L^2_{t,x}$-based estimate for the charge density. (This would correspond, w.r.t. scaling, to a sharp Strichartz estimate in 2D.) More interestingly, one may derive from our result Bourgain's bilinear improvement ([3]).
- All our identities apply to nonlinear equations, and have bilinear versions.
- Nothing but integration by parts is used in the proof: as such, these estimates extend to domains, provided one may control the boundary terms; in the case of Dirichlet boundary conditions, such control is provided by local smoothing.
- As an application to exterior domains, we improve the well-posedness theory to H^1-subcritical (subquintic) nonlinearities for $n = 3$.
- Applications to scattering problems are straightforward, and this extends to 3D exterior domains, where no results were available to our knowledge and where we obtain scattering in the energy class for the defocusing cubic equation.

While presenting this work at Oberwolfach, we learned that similar results (namely a priori bound (2.9)) have been obtained simultaneously and independently by J. Colliander, M. Grillakis and N. Tzirakis, see [9] and [10]), through a different derivation.

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2. Main results

2.1. The Schrödinger equation in $\mathbb{R}^n$

Let $n \geq 1$, $p \in \mathbb{R}$, $p \geq 1$, $\varepsilon \in \{-1, 0, 1\}$, and u is a solution to

$$i\partial_t u + \Delta u = \varepsilon |u|^{p-1} u, \text{ with } u_{|t=0} = u_0. \tag{2.1}$$

We will also need v, solution to

$$i\partial_t v + \Delta v = \varepsilon |v|^{p-1} v, \text{ with } v_{|t=0} = v_0. \tag{2.2}$$

Let us define several quantities which will play a key role: for $n > 1$ and given a function f, its Radon transform is

$$R(f)(s,\omega) = \int_{x\cdot\omega = s} f \, d\mu_{s,\omega}, \tag{2.3}$$

where $\mu_{s,\omega}$ is the induced measure on the hyperplane $x \cdot \omega = s$. We set

$$I_\omega(\varepsilon, u, v) = \int_{x\cdot\omega > y\cdot\omega} (x\cdot\omega - y\cdot\omega)|u|^2(x)|v|^2(y)\, dxdy. \tag{2.4}$$

Remark that a simple computation leads to

(2.5)

$$\partial_t I_\omega = i\left(\int_{x\cdot\omega>y\cdot\omega} \omega\cdot\left[(u\nabla_x\bar u - \bar u \nabla_x u)(x)|v(y)|^2 - (v\nabla_y \bar v - \bar v\nabla_y v)(y)|u(x)|^2\right]\, dy\, dx\right).$$

We may now state our first result.

THEOREM 2.1. – *Let $\omega \in \mathbb{R}^n$, $n > 1$, with $|\omega| = 1$, u solution to (2.1). Then, with $x = x^\perp + s\omega$*

$$\int_s |\partial_s(R(|u|^2))(s,\omega)|^2\, ds + \varepsilon\frac{p-1}{p+1}\int_s R(|u|^2)R(|u|^{p+1})\, ds \tag{2.6}$$
$$+ \int_s\int_{x^\perp\cdot\omega=0}\int_{y^\perp\cdot\omega=0} |u(x^\perp+s\omega)\partial_s u(y^\perp+s\omega) - u(y^\perp+s\omega)\partial_s u(x^\perp+s\omega)|^2\, dx^\perp dy^\perp ds$$
$$= \frac14 \partial_t^2 I_\omega(\varepsilon,u,u).$$

In other words, $I_\omega(\varepsilon,u,u)$ is a convex function in time.

In the specific 1D case, one has actually the following identity.

THEOREM 2.2. – *Let $n = 1$, u, v two solutions to (2.1), (2.2), then*

$$4\int_x |\partial_x(u\bar v)|^2\, dx + 2\varepsilon\frac{p-1}{p+1}\int_x |u|^2|v|^{p+1} + |v|^2|u|^{p+1}\, dx = \partial_t^2 I(\epsilon,u,v). \tag{2.7}$$

REMARK 2.1. – Up to a doubling factor, I_ω may be recast as a Morawetz interaction functional (as introduced in [12]),

$$\int \rho(x-y)|u|^2(x)|v|^2(y)\, dxdy,$$

with $\rho(x-y) = |x\cdot\omega - y\cdot\omega|$. Hence we have replaced the physical distance $|x-y|$ (which was the default choice in [12] and subsequent works) by its projection over a specified direction ω. We chose our definition of I_ω as to emphasize trace terms which will later appear in the proof. In fact, we were led to I_ω by considering variations on the local smoothing, and we will come back to this point in Section 4.2.

In order to turn these bounds into useful nonlinear control, we use

PROPOSITION 2.2. – *Let ω be fixed, then*

$$|\partial_t I_\omega| \leq \|u\|^2_{L^2_x}\|v\|^2_{\dot H^{\frac12}} + \|v\|^2_{L^2_x}\|u\|^2_{\dot H^{\frac12}}. \tag{2.8}$$

As a consequence, when $\varepsilon = 1$ (defocusing equation), we have an priori bound,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} ||\nabla|^{\frac{3-n}{2}}(|u|^2)|^2\,dxdt + \int_{\mathbb{R}}\int_{\mathbb{R}^n} ||\nabla|^{\frac{1-n}{2}}(|u|^{\frac{p+3}{2}})|^2\,dxdt \lesssim sup_{t\in\mathbb{R}}\|u\|^2_{L^2_x}\|u\|^2_{\dot H^{\frac12}}. \tag{2.9}$$

REMARK 2.3. – The right-hand side of (2.8) is very clearly not invariant by galilean transforms. The left-hand side, however, is.

REMARK 2.4. – The a priori estimate (2.9) was obtained simultaneously and independently by J. Colliander, M. Grillakis and N. Tzirakis [9, 10], through a direct derivation with the weight $\rho(x) = |x|$ but with a new commutator argument involving $[x, \sqrt{-\Delta}^{-(n-1)}]$ and the local conservation laws for mass and momentum densities, overcoming the restriction to dimensions $n \geq 3$ from [12].

We now state a more general result: let

$$I_\rho(u,v) = \int \rho(x-y)|u|^2(x)|v|^2(y)\,dxdy. \tag{2.10}$$

Then

THEOREM 2.3. – *Let ρ be a weight function such that its Hessian H_ρ is positive; let*

$$F(u,v)(x,y) = \bar v(y)\nabla_x u(x) + u(x)\nabla_y \bar v(y) \text{ and } G(u,v)(x,y) = v(y)\nabla_x u(x) - u(x)\nabla_y v(y). \tag{2.11}$$

We have

$$\begin{aligned}\partial_t^2 I_\rho = {} & 4\int H_\rho(x-y)(F(u,v)(x,y),\overline{F}(u,v)(x,y))\,dxdy\\ & + \varepsilon\frac{p-1}{p+1}\int |v|^2(y)(\Delta_x\rho)(x-y)|u|^{p+1}(x)\,dxdy\\ & + \varepsilon\frac{p-1}{p+1}\int |u|^2(x)(\Delta_x\rho)(x-y)|v|^{p+1}(y)\,dxdy.\end{aligned} \tag{2.12}$$

Moreover, we may rewrite

$$\int H_\rho(x-y)(F(u,v)(x,y),\overline{F}(u,v)(x,y))\,dxdy = \tag{2.13}$$

$$\int H_\rho(x-y)(G(u,v)(x,y),\overline{G}(u,v)(x,y))\,dxdy + \int \Delta\rho(x-y)\nabla_x(|u|^2(x))\cdot\nabla_y(|v|^2(y)))\,dxdy.$$

REMARK 2.5. – Notice that if we make $u = v$ in (2.13) and assume that the Fourier transform of $\Delta\rho$ is positive, we can bound each of the two terms in the right-hand side in terms of the left-hand side.

The above remark used in the particular case $\rho(z) = |z\cdot\omega|$ gives us the following corollary for the linear equation.