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Vladimir V. FOCK & Alexander B. GONCHAROV

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CLUSTER ENSEMBLES, QUANTIZATION AND THE DILOGARITHM

BY VLADIMIR V. FOCK AND ALEXANDER B. GONCHAROV

ABSTRACT. – A cluster ensemble is a pair $(\mathcal{X}, \mathcal{A})$ of positive spaces (i.e. varieties equipped with positive atlases), coming with an action of a symmetry group Γ. The space $\mathcal{A}$ is closely related to the spectrum of a cluster algebra [12]. The two spaces are related by a morphism $p : \mathcal{A} \longrightarrow \mathcal{X}$. The space $\mathcal{A}$ is equipped with a closed 2-form, possibly degenerate, and the space $\mathcal{X}$ has a Poisson structure. The map p is compatible with these structures. The dilogarithm together with its motivic and quantum avatars plays a central role in the cluster ensemble structure. We define a non-commutative q-deformation of the $\mathcal{X}$-space. When q is a root of unity the algebra of functions on the q-deformed $\mathcal{X}$-space has a large center, which includes the algebra of functions on the original $\mathcal{X}$-space.

The main example is provided by the pair of moduli spaces assigned in [6] to a topological surface S with a finite set of points at the boundary and a split semisimple algebraic group G. It is an algebraic-geometric avatar of higher Teichmüller theory on S related to G.

We suggest that there exists a duality between the $\mathcal{A}$ and $\mathcal{X}$ spaces. In particular, we conjecture that the tropical points of one of the spaces parametrise a basis in the space of functions on the Langlands dual space. We provide some evidence for the duality conjectures in the finite type case.

RÉSUMÉ. – Un ensemble amassé est une paire $(\mathcal{X}, \mathcal{A})$ d'espaces positifs (i.e. de variétés munies d'un atlas positif) munis de l'action d'un groupe discret. L'espace $\mathcal{A}$ est relié au spectre d'une algèbre amassée [12]. Les deux espaces sont liés par un morphisme $p : \mathcal{A} \longrightarrow \mathcal{X}$. L'espace $\mathcal{A}$ est muni d'une 2-forme fermée, éventuellement dégénérée, et l'espace $\mathcal{X}$ est muni d'une structure de Poisson. L'application p est compatible avec ces structures. Le dilogarithme avec ses avatars motiviques et quantiques joue un rôle fondamental dans la structure d'un ensemble amassé. Nous définissons une déformation non-commutative de l'espace $\mathcal{X}$. Nous montrons que, dans le cas où le paramètre de la déformation q est une racine de l'unité, l'algèbre déformée a un centre qui contient l'algèbre des fonctions sur l'espace $\mathcal{X}$ originel.

Notre exemple principal est celui de l'espace des modules associé dans [6] à une surface topologique S munie d'un nombre fini de points distingués sur le bord et à un groupe algébrique semi-simple G. C'est un avatar algébro-géométrique de la théorie de Teichmüller d'ordre supérieur sur la surface S à valeurs dans G.

Nous évoquons l'existence d'une dualité entre les espaces $\mathcal{A}$ et $\mathcal{X}$. Une des manifestations de cette dualité est une conjecture de dualité affirmant que les points tropicaux d'un espace paramètrent une base dans l'espace d'une certaine classe de fonctions sur l'espace Langlands-dual. Nous démontrons cette conjecture dans un certain nombre d'exemples.

Contents

1. Introduction and main definitions with simplest examples

Cluster algebras are a remarkable discovery of S. Fomin and A. Zelevinsky [12]. They are certain commutative algebras defined by a very simple and general data.

We show that a cluster algebra is part of a richer structure, which we call a *cluster ensemble*. A cluster ensemble is a pair $(\mathcal{X}, \mathcal{A})$ of *positive spaces* (which are varieties equipped with positive atlases), coming with an action of a certain discrete symmetry group Γ. These two spaces are related by a morphism $p : \mathcal{A} \longrightarrow \mathcal{X}$, which in general, as well as in many interesting examples, is neither injective nor surjective. The space $\mathcal{A}$ has a degenerate symplectic structure, and the space $\mathcal{X}$ has a Poisson structure. The map p relates the Poisson and degenerate symplectic structures in a natural way. Amazingly, the dilogarithm together with its motivic and quantum avatars plays a central role in the cluster ensemble structure. The space $\mathcal{A}$ is closely related to the spectrum of a cluster algebra. On the other hand, in many situations the most interesting part of the structure is the space $\mathcal{X}$.

We define a canonical non-commutative q-deformation of the $\mathcal{X}$-space. We show that when q is a root of unity the algebra of functions on the q-deformed $\mathcal{X}$-space has a large center, which contains a subalgebra identified with the algebra of functions on the original $\mathcal{X}$-space.

The main example, as well as the main application of this theory so far, is provided by the $(\mathcal{X}, \mathcal{A})$-pair of moduli spaces assigned in [6] to a topological surface S with a finite set of points at the boundary and a semisimple algebraic group G. In particular, the $\mathcal{X}$-space in the simplest case when $G = PGL_2$ and S is a disc with n points at the boundary is the moduli space $\mathcal{M}_{0,n}$.

This pair of moduli spaces is an algebraic-geometric avatar of higher Teichmüller theory on S related to G. In the case $G = SL_2$ we get the classical Teichmüller theory, as well as its generalization to surfaces with a finite set of points on the boundary. A survey of the Teichmüller theory emphasizing the cluster point of view can be found in [7].

We suggest that there exists a duality between the $\mathcal{A}$ and $\mathcal{X}$ spaces. One of its manifestations is our package of duality conjectures in Section 4. These conjectures assert that the *tropical points* of the $\mathcal{A}/\mathcal{X}$-space parametrise a basis in a certain class of functions on the *Langlands dual* $\mathcal{X}/\mathcal{A}$-space. It can be viewed as a canonical function (the *universal kernel*) on the product of the set of tropical points of one space and the Langlands dual space.

To support these conjectures, we define in Section 5.1 the tropical limit of such a universal kernel in the finite type case. Another piece of evidence is provided by Chapter 12 in [6].

In the rest of the introduction we define cluster $\mathcal{X}$- and $\mathcal{A}$-varieties and describe their key features. Section 1.1 provides background on positive spaces, borrowed from Chapter 4 of [6]. Cluster varieties are defined in Section 1.2. In Section 1.3 we discuss one of the simplest examples: cluster $\mathcal{X}$-variety structures of the moduli space $\mathcal{M}_{0,n+3}$. In Section 1.4 we summarize the main structures of cluster varieties. In Section 1.4 we discuss how they appear in our main example: higher Teichmüller theory.

1.1. Positive schemes and positive spaces

A *semifield* is a set P equipped with the operations of addition and multiplication, so that addition is commutative and associative, multiplication makes P an abelian group, and they are compatible in a usual way: $(a + b)c = ac + bc$ for $a, b, c \in P$. A standard example is given by the set $\mathbb{R}_{>0}$ of positive real numbers. Here are more exotic examples. Let $\mathbb{A}$ be one of the sets $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. The *tropical semifield* $\mathbb{A}^t$ associated with $\mathbb{A}$ is the set $\mathbb{A}$ with the multiplication $\otimes$ and addition $\oplus$ given by

$$a \otimes b := a + b, \quad a \oplus b := \max(a, b).$$

One more example is given by the semifield $\mathbb{R}_{>0}((\varepsilon))$ of Laurent series in ε with real coefficients and a positive leading coefficient, equipped with the usual addition and multiplication. There is a homomorphism of semifields $-\deg : \mathbb{R}_{>0}((\varepsilon)) \to \mathbb{Z}^t$, given by $f \longmapsto -\deg(f)$. It explains the origin of the tropical semifield $\mathbb{Z}^t$.

Recall the standard notation $\mathbb{G}_m$ for the multiplicative group. It is an affine algebraic group. The ring of regular functions on $\mathbb{G}_m$ is $\mathbb{Z}[X, X^{-1}]$, and for any field F one has $\mathbb{G}_m(F) = F^*$. A product of multiplicative groups is known as a *split algebraic torus* over $\mathbb{Z}$, or simply a split algebraic torus.

Let H be a split algebraic torus. A rational function f on H is called *positive* if it belongs to the semifield generated, in the field of rational functions on H, by the characters of H. So it can be written as $f = f_1/f_2$ where f_1, f_2 are linear combinations of characters with positive integral coefficients. A *positive rational map* between two split tori H_1, H_2 is a rational map $f : H_1 \to H_2$ such that f^* induces a homomorphism of the semifields of positive rational functions. Equivalently, for any character χ of H_2 the composition $\chi \circ f$ is a positive rational function on H_1. A composition of positive rational functions is positive. Let Pos be the category whose objects are split algebraic tori and morphisms are positive rational maps. A *positive divisor* in a torus H is a divisor given by an equation $f = 0$, where f is a positive rational function on H.

Definition 1.1. – *A positive atlas on an irreducible scheme/stack X over $\mathbb{Q}$ is a family of birational isomorphisms*

$$\psi_\alpha : H_\alpha \longrightarrow X, \quad \alpha \in \mathcal{C}_X, \tag{1}$$

between split algebraic tori H_α and X, parametrised by a non empty set $\mathcal{C}_X$, such that:

i) *each ψ_α is regular on the complement of a positive divisor in H_α;*

ii) *for any $\alpha, \beta \in \mathcal{C}_X$ the map $\psi_{\alpha,\beta} := \psi_\beta^{-1} \circ \psi_\alpha : H_\alpha \longrightarrow H_\beta$ is a positive rational map* (1).

A positive atlas is called regular if each ψ_α is regular.

Birational isomorphisms (1) are called positive coordinate systems on X. A *positive scheme* is a scheme equipped with a positive atlas. We will need an equivariant version of this definition.

Definition 1.2. – *Let Γ be a group of automorphisms of X. A positive atlas* (1) *on X is Γ-equivariant if Γ acts on the set $\mathcal{C}_X$, and for every $\gamma \in \Gamma$ there is an isomorphism of algebraic tori $i_\gamma : H_\alpha \xrightarrow{\sim} H_{\gamma(\alpha)}$ making the following diagram commutative:*

$$\begin{array}{ccc} H_\alpha & \xrightarrow{\psi_\alpha} & X \\ \downarrow i_\gamma & & \downarrow \gamma \\ H_{\gamma(\alpha)} & \xrightarrow{\psi_{\gamma(\alpha)}} & X. \end{array} \tag{2}$$

(1) A positive atlas covers a non-empty Zariski open subset of X, but not necessarily the whole space X.