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ON THE DE RHAM AND p-ADIC REALIZATIONS OF THE ELLIPTIC POLYLOGARITHM FOR CM ELLIPTIC CURVES

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ABSTRACT. – In this paper, we give an explicit description of the de Rham and p-adic polylogarithms for elliptic curves using the Kronecker theta function. In particular, consider an elliptic curve E defined over an imaginary quadratic field $\mathbb{K}$ with complex multiplication by the full ring of integers $\mathcal{O}_{\mathbb{K}}$ of $\mathbb{K}$. Note that our condition implies that $\mathbb{K}$ has class number one. Assume in addition that E has good reduction above a prime $p \geq 5$ unramified in $\mathcal{O}_{\mathbb{K}}$. In this case, we prove that the specializations of the p-adic elliptic polylogarithm to torsion points of E of order prime to p are related to p-adic Eisenstein-Kronecker numbers. Our result is valid even if E has supersingular reduction at p. This is a p-adic analogue in a special case of the result of Beilinson and Levin, expressing the Hodge realization of the elliptic polylogarithm in terms of Eisenstein-Kronecker-Lerch series. When p is ordinary, then we relate the p-adic Eisenstein-Kronecker numbers to special values of p-adic L-functions associated to certain Hecke characters of $\mathbb{K}$.

RÉSUMÉ. – Dans cet article, nous donnons une description explicite des réalisations de de Rham et p-adiques des polylogarithmes elliptiques en utilisant la fonction thêta de Kronecker. Considérons en particulier une courbe elliptique E définie sur un corps quadratique imaginaire $\mathbb{K}$, à multiplication complexe par l'anneau des entiers $\mathcal{O}_{\mathbb{K}}$ de $\mathbb{K}$, et ayant bonne réduction en chaque place au-dessus d'un nombre premier $p \geq 5$ non ramifié dans $\mathbb{K}$. On notera que le nombre de classe de $\mathbb{K}$ est nécessairement égal à un. Nous montrons alors que les spécialisations des polylogarithmes p-adiques aux points de torsion de E d'ordre premier à p sont reliées aux nombres d'Eisenstein-Kronecker p-adiques. Ce résultat est valable même si E a une réduction supersingulière en p. C'est un analogue p-adique d'un cas spécial du résultat de Beilinson et Levin exprimant la réalisation de Hodge du polylogarithme elliptique en utilisant les séries d'Eisenstein-Kronecker-Lerch. Si p est quelconque, nous établissons un lien entre les nombres d'Eisenstein-Kronecker p-adiques et les valeurs spéciales des fonctions L associées aux caractères de Hecke de $\mathbb{K}$.

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0. Introduction

0.1. Introduction

In the paper [7], Beilinson and Levin constructed the elliptic polylogarithm, which is an element in absolute Hodge or ℓ-adic cohomology of an elliptic curve minus the identity. This construction is a generalization to the case of elliptic curves of the construction by Beilinson and Deligne of the polylogarithm sheaf on the projective line minus three points. The purpose of this paper is to study the p-adic realization of the elliptic polylogarithm for an elliptic curve with complex multiplication, even when the elliptic curve has supersingular reduction at the prime p.

To achieve our goal, we first describe the de Rham realization of the elliptic polylogarithm for a general elliptic curve defined over a subfield of $\mathbb{C}$. In particular, we explicitly describe the connection of the elliptic polylogarithm using rational functions. Similar results were obtained by Levin and Racinet [21] Section 5.1.3, and Besser and Solomon [28].

The de Rham realization of the elliptic polylogarithm gives the coherent module with connection underlying the polylogarithm sheaf in the Hodge and p-adic cases. We construct the p-adic realization of the elliptic polylogarithm as a filtered overconvergent F-isocrystal on the elliptic curve minus the identity, when the elliptic curve is defined over an imaginary quadratic field, has complex multiplication by its ring of full integers, and has good reduction over a fixed prime $p \geq 5$ unramified in the ring of complex multiplication. The Frobenius structure on the p-adic elliptic polylogarithm by definition is compatible with the connection of the underlying de Rham realization. Our main result, Theorem 4.19, is an explicit description of the Frobenius structure on the p-adic elliptic polylogarithm sheaf in terms of overconvergent functions characterized as the solutions of the p-adic differential equations arising from the compatibility of the Frobenius with the connection.

Using this description, we calculate the specializations of the p-adic elliptic polylogarithm to torsion points of order prime to p (more precisely, torsion points of order prime to $\mathfrak{p}$) and prove that the specializations give the p-adic Eisenstein-Kronecker numbers, which are special values of the p-adic distribution interpolating Eisenstein-Kronecker numbers (see Theorem 5.7). This result is a generalization of the result of [4], where we have dealt only with the one variable case for an ordinary prime. A similar result concerning the specialization in two-variables was obtained in [5], again for ordinary primes, using a very different method. The result of the current paper is valid even when p is supersingular.

When the elliptic curve has good ordinary reduction at the primes above p, the p-adic Eisenstein-Kronecker numbers are related to special values of p-adic L-functions which p-adically interpolate special values of certain Hecke L-functions associated to imaginary quadratic fields (see Proposition 2.27). Hence our main result in the ordinary case implies that the specialization of the p-adic elliptic polylogarithm to torsion points as above are related to special values of certain p-adic L-functions (Corollary 5.10). Recently, Solomon [28] has announced that the p-adic elliptic polylogarithm as constructed in this paper is the image by the syntomic regulator of the motivic elliptic polylogarithm. Assuming this fact, our result may be interpreted as a p-adic analogue of Beilinson's conjecture.

In the appendix, modeling on our approach of the p-adic case, we calculate the real Hodge realization of the elliptic polylogarithm by solving certain iterated differential equations as

in the p-adic case. The Hodge realization of the elliptic polylogarithm was first described by Beilinson-Levin [7] and Wildeshaus [32]. We give an alternative description of the real Hodge realization in terms of multi-valued meromorphic functions given as the solutions of these differential equations. Our method highlights the striking similarity between the classical and the p-adic cases.

0.2. Overview

The detailed content of this paper is as follows. In §1, we introduce the Kronecker theta function $\Theta(z,w)$, which is our main tool in describing the elliptic polylogarithm. A slightly modified version of this function was previously used by Levin [20] to describe the analytic aspect of the elliptic polylogarithm. We use this function to construct rational functions L_n on the elliptic curve, which we call the connection functions. The main result of the first section is the explicit description of the de Rham realization of the elliptic polylogarithm in terms of L_n (Corollary 1.42).

The main result of this paper is an explicit description of the p-adic elliptic polylogarithm for CM elliptic curves. Let $\mathbb{K}$ be an imaginary quadratic field, and let E be an elliptic curve defined over $\mathbb{K}$ with complex multiplication by the full ring of integers $\mathcal{O}_{\mathbb{K}}$ of $\mathbb{K}$. Note that by the theory of complex multiplication, the existence of E implies that the class number of $\mathbb{K}$ is one. We assume in addition that E has good reduction above a prime $p \geq 5$ unramified in $\mathcal{O}_{\mathbb{K}}$. We denote by $\psi_{E/\mathbb{K}}$ the Grössencharacter of E over $\mathbb{K}$. We fix a prime $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{K}}$ over p, and we let $\pi := \psi_{E/\mathbb{K}}(\mathfrak{p})$. Let Γ be the period lattice of E for some invariant differential ω defined over $\mathcal{O}_{\mathbb{K}}$.

In §2, we introduce the Eisenstein-Kronecker-Lerch series and Eisenstein-Kronecker numbers. We fix a lattice Γ in $\mathbb{C}$. Let $z_0 \in \mathbb{C} \setminus \Gamma$. We define the Eisenstein-Kronecker numbers $e^*_{a,b}(z_0)$ for integers a and b by the formula

$$e^*_{a,b}(z_0) = \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{\overline{\gamma}^a}{\gamma^b} \langle \gamma, z_0 \rangle,$$

where $\langle \gamma, z_0 \rangle := \exp((\gamma \overline{z}_0 - z_0 \overline{\gamma})/A)$ and A is the fundamental area of Γ divided by $\varpi = 3.14159\cdots$. The above sum converges only for $b > a+2$, but one may give it meaning for all a and b by analytic continuation. Let z_0 be a point in $\mathbb{C} \setminus \Gamma$ which defines a non-zero torsion point in $E(\overline{\mathbb{Q}}) \cong \mathbb{C}/\Gamma$, which we again denoted by z_0. By Damerell's theorem, the numbers $e^*_{a,b}(z_0)/A^a$ are algebraic over $\mathbb{K}$ when $a, b \geq 0$. We fix once and for all an embedding $i_{\mathfrak{p}} : \overline{\mathbb{K}} \hookrightarrow \mathbb{C}_p$ continuous for the $\mathfrak{p}$-adic topology on $\mathbb{K}$, and we regard $e^*_{a,b}(z_0)/A^a$ for $a, b \geq 0$ as p-adic numbers through this embedding. The Hodge realization of the elliptic polylogarithm is related to Eisenstein-Kronecker numbers $e^*_{a,b}(z_0)$ for $a < 0$ (see Theorem A.29), which are complex numbers expected to be transcendental. We use p-adic interpolation to define p-adic versions of $e^*_{a,b}(z_0)$ for $a < 0$.

Assume now that p is ordinary of the form $(p) = \mathfrak{p}\mathfrak{p}^*$ in $\mathcal{O}_{\mathbb{K}}$, and suppose that z_0 is a non-zero torsion point of order prime to p. For their construction of the two-variable p-adic L-function for CM elliptic curves (see also [6]), Manin-Vishik [22] and Katz [19] constructed

a p-adic measure $\mu_{z_0,0}$ on $\mathbb{Z}_p \times \mathbb{Z}_p$ satisfying

$$\frac{1}{\Omega_{\mathfrak{p}}^{a+b}} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,0}(x,y) = (-1)^{a+b} \left(\frac{e^*_{a,b+1}(z_0)}{A^a} - \frac{\pi^a e^*_{a,b+1}(\pi z_0)}{\overline{\pi}^{b+1} A^a} \right)$$

for any $a, b \geq 0$, where $\Omega_{\mathfrak{p}}$ is a certain p-adic period in $W^\times$ for the ring of integers W of the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. Using this measure, we define the p-adic Eisenstein-Kronecker numbers as follows.

DEFINITION 0.1. – Suppose z_0 is a non-zero torsion point of $E(\overline{\mathbb{Q}})$ of order prime to p. For any integer a, b such that $b \geq 0$, we define the p-adic Eisenstein-Kronecker number $e^{(p)}_{a,b+1}(z_0)$ by the formula

$$e^{(p)}_{a,b+1}(z_0) := \frac{1}{\Omega_{\mathfrak{p}}^{a+b} b!} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^a y^b d\mu_{z_0,0}(x,y).$$

Note that this definition is valid even for $a < 0$.

If p is supersingular, which due to our assumption that p be unramified in $\mathcal{O}_{\mathbb{K}}$ is equivalent to the condition that p remains prime in $\mathcal{O}_{\mathbb{K}}$, then a two-variable measure as above interpolating Eisenstein-Kronecker numbers does not exist. We define $e^{(p)}_{a,b+1}(z_0)$ using p-adic distributions, constructed originally by Boxall [12][11], Schneider-Teitelbaum [25], Fourquaux and Yamamoto [33], which interpolate in one-variable Eisenstein-Kronecker numbers for fixed $b \geq 0$. The latter construction is valid even when p is ordinary, and the definition in this case is equivalent to the one given above. In both constructions, the fact that the generating function for Eisenstein-Kronecker numbers is given by the Kronecker theta function $\Theta_{z_0,w_0}(z,w)$ ([6] Theorem 1.17) is crucial.

In §3, we give the definition of the p-adic elliptic polylogarithm functions, which are overconvergent functions on the elliptic curve minus the residue disc around the identity characterized as the solutions of a certain differential equation. We then give the relation of these functions to the p-adic Eisenstein-Kronecker numbers.

In §4, we construct and explicitly calculate the p-adic elliptic polylogarithm. Let K be a finite unramified extension of $\mathbb{K}_{\mathfrak{p}}$. We denote by $\mathcal{O}_K$ the ring of integers of K and by k its residue field. We denote again by E a model of our elliptic curve over $\mathcal{O}_K$. The rigid cohomology $H^1_{\mathrm{rig}}(E_k/K)$ of $E_k := E \otimes k$ is a Frobenius K-module with Hodge filtration coming from the Hodge filtration of de Rham cohomology of $E_K := E \otimes K$ through the canonical isomorphism

$$H^1_{\mathrm{dR}}(E_K/K) \cong H^1_{\mathrm{rig}}(E_k/K).$$

This cohomology group is a K-vector space with certain basis $\underline{\omega}$ and $\underline{\omega}^*$. We let $\mathscr{H}$ be the filtered Frobenius module dual to $H^1_{\mathrm{rig}}(E_k/K)$, and we denote by $\underline{\omega}^\vee$ and $\underline{\omega}^{*\vee}$ the dual basis.

Let $S(E)$ be the category of filtered overconvergent F-isocrystals on E, referred to as the category of syntomic coefficients in our previous papers, which plays a rough p-adic analogue of the category of variations of mixed Hodge structures on E. We denote by $S(\mathscr{V})$ the same category on $\mathscr{V} := \mathrm{Spec}\, \mathcal{O}_K$, which is simply the category of filtered Frobenius modules. The elliptic logarithm sheaf $\mathscr{Log}$ is a pro-object $\mathscr{Log} = \varprojlim \mathscr{Log}^N$ in $S(E)$. One of its main features is the splitting principle, given as follows.