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# *Explicit birational geometry of threefolds of general type, I*

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# EXPLICIT BIRATIONAL GEOMETRY OF THREEFOLDS OF GENERAL TYPE, I

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ABSTRACT. – Let $V$ be a complex nonsingular projective 3-fold of general type. We prove $P_{12}(V) := \dim H^0(V, 12K_V) > 0$ and $P_{m_0}(V) > 1$ for some positive integer $m_0 \leq 24$. A direct consequence is the birationality of the pluricanonical map $\varphi_m$ for all $m \geq 126$. Besides, the canonical volume Vol$(V)$ has a universal lower bound $\nu(3) \geq \frac{1}{63 \cdot 126^2}$.

RÉSUMÉ. – Soit $V$ une variété non singulière complexe de type général et de dimension 3. Nous montrons $P_{12}(V) := \dim H^0(V, 12K_V) > 0$ et $P_{m_0}(V) > 1$ pour un certain entier $m_0 \leq 24$. Une conséquence directe est la birationalité de l'application pluricanonique $\varphi_m$ pour tout $m \geq 126$. De plus, le volume canonique Vol$(V)$ a un minorant universel $\nu(3) \geq \frac{1}{63 \cdot 126^2}$.

## 1. Introduction

Let $Y$ be a nonsingular projective variety of dimension $n$. It is said to be of general type if the pluricanonical map $\varphi_m := \Phi_{|mK_Y|}$ corresponding to the linear system $|mK_Y|$ is birational into a projective space for $m \gg 0$. Thus it is natural and important to ask:

PROBLEM 1. – Can one find a constant $c(n)$, so that $\varphi_m$ is birational onto its image for all $m \geq c(n)$ and for all $Y$ with $\dim Y = n$?

When $\dim Y = 1$, it was classically known that $|mK_Y|$ gives an embedding of $Y$ into a projective space if $m \geq 3$. When $\dim Y = 2$, Kodaira-Bombieri's theorem [2] says that $|mK_Y|$ gives a birational map onto the image for $m \geq 5$. This theorem has essentially established the canonical classification theory for surfaces of general type.

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A natural approach to study this problem in higher dimensions is an induction on the dimension by utilizing vanishing theorems. This amounts to estimating the plurigenus, for which purpose the greatest difficulty seems to be to bound from below the canonical volume

$$\mathrm{Vol}(Y) := \limsup_{\{m\in\mathbb{Z}^+\}}\{\frac{n!}{m^n}\dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(mK_Y))\}.$$

The volume is an integer when $\dim Y \leq 2$. However it is only a rational number in general, which may account for the complexity of high dimensional birational geometry. In fact, it is almost an equivalent question to study the lower bound of the canonical volume.

Problem 2. – Can one find a constant $\nu(n)$ such that $\mathrm{Vol}(Y) \geq \nu(n)$ for all varieties $Y$ of general type with $\dim Y = n$?

A recent result of Hacon and M$^{c}$Kernan [13], Takayama [24] and Tsuji [25] shows the existence of both $c(n)$ and $\nu(n)$. An explicit constant $c(n)$ or $\nu(n)$ is, however, mysterious at least up to now. Notice that similar questions were asked by Kollár and Mori [19, 7.74].

Here we mainly deal with $c(3)$ and $\nu(3)$. For known results under extra assumptions, one may refer to [3, 4, 5, 6, 8, 9, 10, 14, 18, 20] and others. In this series of papers, we would like to present two realistic constants $c(3)$ and $\nu(3)$. In fact, our method can help us to prove some sharp results. Being worried that a very long paper would tire the readers, we decided to only explain our key technique and rough statements in the first part whereas more refined and some sharp statements will be presented in the subsequent papers. Our main result in this paper is the following:

Theorem 1.1. – *Let $V$ be a nonsingular projective 3-fold of general type. Then*

(1) $P_{12} > 0$*;*
(2) $P_{m_0} \geq 2$ *for some positive integer* $m_0 \leq 24$*.*

With Kollár's result [18, Corollary 4.8] and its improved form [7, Theorem 0.1], we immediately get the following:

Corollary 1.2. – *Let $V$ be a nonsingular projective 3-fold of general type. Then*

(1) $\varphi_m$ *is birational onto its image for all* $m \geq 126$*.*
(2) *Vol*$(V) \geq \frac{1}{63\cdot 126^2}$*.*

Example 1.3 (see [14, p. 151, No. 23]). – The "worst" known example is a general weighted hypersurface $X = X_{46} \subset \mathbb{P}(4,5,6,7,23)$. The 3-fold $X$ has invariants: $p_g(X) = P_2(X) = P_3(X) = 0$, $P_4(X) = \cdots = P_9(X) = 1$, $P_{10}(X) = 2$ and $\mathrm{Vol}(X) = \frac{1}{420}$. Moreover, it is known that $\varphi_m$ is birational for all $m \geq 27$, but $\varphi_{26}$ is not birational.

Now we explain the main idea of our paper. It is very natural to investigate the plurigenus $P_m$, which can be calculated using Reid's Riemann-Roch formula in [21, 23]. However the most difficult point is to control the contribution from singularities due to the combinatorial complexity of baskets of singularities on the 3-fold.

Indeed, given a minimal 3-fold $X$ with at worst canonical singularities, a known fact is that the canonical volume and all plurigenera are determined by the basket (of singularities) $B$, $\chi = \chi(\mathcal{O}_X)$ and $P_2 = P_2(X)$. We call the triple $(B, \chi, P_2)$ a *formal basket*. First we will define a partial ordering (called "packing") between formal baskets. (In this paper, we are only concerned about the numerical behavior of "packing", rather than its geometric meaning. More details on its geometric aspect will be explored in our subsequent works.) Then we introduce the "canonical sequence of prime unpackings of a basket"

$$B^{(0)} \succcurlyeq B^{(5)} \succcurlyeq ... \succcurlyeq B^{(n)} \succcurlyeq ... \succcurlyeq B$$

and, furthermore, each step in the sequence can be calculated in terms of the datum of the given formal basket. The intrinsic properties of the canonical sequence tell us many new inequalities among the Euler characteristic and the plurigenus, of which the most interesting one is:

$$2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13}.$$

If $P_{m_0} \geq 2$ for some $m_0 \leq 12$, then one gets many interesting results by [18, Corollary 4.8] and [7, Theorem 0.1]. Otherwise one has $P_m \leq 1$ for all $m \leq 12$ and the above inequality gives $\chi \leq 8$. This essentially tells us that the number of formal baskets is finite! Thus, theoretically, we are able to obtain various effective results.

Here is the overview to the structure of this paper. In Section 2, we introduce the notion of packing and define some invariants of baskets. Then we define the canonical sequence of "prime unpackings" of a basket and give some examples. In Section 3, we define the notion of formal baskets. Then we study various relations among formal baskets, Euler characteristics and $K^3$. We calculate the first few elements in the canonical sequence of the given basket. This immediately gives many inequalities among Euler characteristics. We would like to remark that the method so far works for $\mathbb{Q}$-factorial threefolds (not only of general type) with canonical singularities. With all these preparations, we prove the main theorem on threefolds of general type in Section 4.

Another remark is that the method in Sections 2 and 3 is also valid for $\mathbb{Q}$-Fano threefolds. More precisely, there are similar relations among formal baskets, anti-plurigenera and the anti-canonical volume with proper sign alterations because of Serre dualities. We will explore some more applications of our method in a future work.

In our next paper of this series, we will work out some classification of formal baskets with given small Euler characteristics. Together with some more detailed study of the geometry of pluricanonical maps, we will prove the following theorem:

THEOREM A. – *Let V be a nonsingular projective 3-fold of general type. Then the following hold.*

(i) *$\varphi_m$ is birational onto its image for all $m \geq 73$.*
(ii) $\mathrm{Vol}(V) \geq \frac{1}{2660}$.
(iii) *Suppose that $\chi(\mathcal{O}_V) \leq 1$. Then $\mathrm{Vol}(V) \geq \frac{1}{420}$, which is optimal. Moreover $\varphi_m$ is birational for all $m \geq 40$.*

Throughout, we work over the complex number field $\mathbb{C}$. We prefer to use $\sim$ to denote the linear equivalence and $\equiv$ means numerical equivalence. We mainly refer to [17, 19, 22] for tool books on 3-dimensional birational geometry.

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## 2. Baskets of singularities

In this section, we introduce the notion of packing between baskets of singularities. This notion defines a partial ordering on the set of baskets. For a given basket, we define its canonical sequence of prime unpackings. The canonical sequence trick is a fundamental and effective tool in our arguments.

2.1. – Terminal quotient singularity and basket. By *a 3-dimensional terminal quotient singularity $Q$ of type* $\frac{1}{r}(1,-1,b)$, we mean a singularity which is analytically isomorphic to the quotient of $(\mathbb{C}^3, \mathrm{o})$ by a cyclic group action $\varepsilon$:

$$\varepsilon(x,y,z)=(\varepsilon x,\varepsilon^{-1}y,\varepsilon^{b}z)$$

where $r$ is a positive integer, $\varepsilon$ is a fixed $r$-th primitive root of 1, the integer $b$ is coprime to $r$ and $0<b<r$.

2.2. – Convention. By replacing $\varepsilon$ with another primitive root of 1 and changing the ordering of coordinates, we may and will assume that $b\leq \frac{r}{2}$.

*A basket $B$* of singularities is a collection (allowing multiplicities) of terminal quotient singularities of type $\frac{1}{r_i}(1,-1,b_i)$, $i\in I$ where $I$ is a finite index set. For simplicity, we will always denote a terminal quotient singularity $\frac{1}{r}(1,-1,b)$ by a pair of integers $(b,r)$. So we will write a basket as:

$$B:=\{n_i\times(b_i,r_i)\mid i\in J,\ n_i\in\mathbb{Z}^+\},$$

where $n_i$ denotes the multiplicities.

Given baskets $B_1=\{n_i\times(b_i,r_i)\}$ and $B_2=\{m_i\times(b_i,r_i)\}$, we define

$$B_1\cup B_2:=\{(n_i+m_i)\times(b_i,r_i)\}.$$

Definition 2.3. – A generalized basket means a collection of pairs of integers $(b,r)$ with $0<b<r$, not necessarily coprime and allowing multiplicities.

2.4. – Invariants of baskets. Given a generalized basket $(b,r)$ with $b\leq\frac{r}{2}$ and a fixed integer $n>0$. Let $\delta:=\lfloor\frac{bn}{r}\rfloor$. Then $\frac{\delta+1}{n}>\frac{b}{r}\geq\frac{\delta}{n}$. We define

$$\Delta^n(b,r):=\delta bn-\frac{(\delta^2+\delta)}{2}r. \tag{2.1}$$

One can see that $\Delta^n(b,r)$ is a non-negative integer. For a generalized basket $B=\{(b_i,r_i)\}_{i\in I}$ and a fixed $n>0$, we define $\Delta^n(B):=\sum_{i\in I}\Delta^n(b_i,r_i)$. By definition, $\Delta^2(B)=0$ for any basket $B$. By a direct calculation, one gets the following relation:

$$\frac{\overline{jb_i}(r_i-\overline{jb_i})}{2r_i}-\frac{jb_i(r_i-jb_i)}{2r_i}=\Delta^j(b_i,r_i)$$