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A CHEN MODEL FOR MAPPING SPACES AND THE SURFACE PRODUCT

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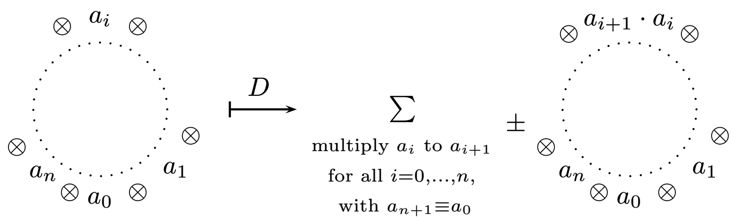
ABSTRACT. – We develop a machinery of Chen iterated integrals for higher Hochschild complexes. These are complexes whose differentials are modeled on an arbitrary simplicial set much in the same way the ordinary Hochschild differential is modeled on the circle. We use these to give algebraic models for general mapping spaces and define and study the surface product operation on the homology of mapping spaces of surfaces of all genera into a manifold. This is an analogue of the loop product in string topology. As an application, we show this product is homotopy invariant. We prove Hochschild-Kostant-Rosenberg type theorems and use them to give explicit formulae for the surface product of odd spheres and Lie groups.

RÉSUMÉ. – Dans cet article, on étend le formalisme des intégrales itérées de Chen aux complexes de Hochschild supérieurs. Ces derniers sont des complexes de (co)chaînes modelés sur un espace (simplicial) de la même manière que le complexe de Hochschild classique est modelé sur le cercle. On en déduit des modèles algébriques pour les espaces fonctionnels que l'on utilise pour étudier le produit surfacique. Ce produit, défini sur l'homologie des espaces de fonctions continues de surfaces (de genre quelconque) dans une variété, est un analogue du produit de Chas-Sullivan sur les espaces de lacets en topologie des cordes. En particulier, on en déduit que le produit surfacique est un invariant homotopique. On démontre également un théorème du type Hochschild-Kostant-Rosenberg pour les complexes de Hochschild modelés sur les surfaces qui permet d'obtenir des formules explicites pour le produit surfacique des sphères de dimension impaire ainsi que pour les groupes de Lie.

1. Introduction

An element of the Hochschild chain complex $CH_\bullet(A, A)$ of an associative algebra A is by definition an element in the multiple tensor product $A \otimes \cdots \otimes A$. When defining the differential $D : CH_\bullet(A, A) \to CH_{\bullet-1}(A, A)$ however, it is instructive to picture this linear sequence of tensor products in a circular configuration, because the differential multiplies any two adjacent tensor factors starting from the beginning until the end and at the very end multiplies the last factor of the sequence with the first factor, as shown below.

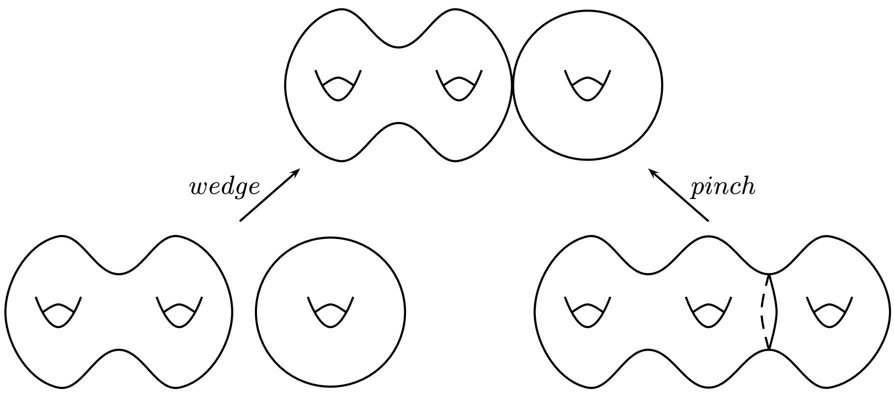


As it turns out this is not just a mnemonic device but rather an explanation of the fundamental connection between the Hochschild chain complex and the circle, which, for instance gives rise to the cyclic structure of the Hochschild chain complex and thus to cyclic homology, see [21]. This connection is also at the heart of the relationship between the Hochschild complex of the differential forms $\Omega^\bullet M$ on a manifold M, and the differential forms $\Omega^\bullet(LM)$ on the free loop space LM of M, which is the space of smooth maps from the circle S^1 to the manifold M; see [14]. At the core of this connection is the fact that the Hochschild complex is the underlying complex of a simplicial module whose simplicial structure is modelled on a particular simplicial model $S^1_\bullet$ of the circle. The principle behind this can be fruitfully used to construct new complexes whose module structure and differential are combinatorially governed by a given simplicial set $X_\bullet$, much in the same way that the ordinary Hochschild complex is governed by $S^1_\bullet$; see [25]. However carrying the construction to higher dimensional simplicial sets turns out to require associative and commutative algebras. The result of these constructions define for any (differential graded) commutative algebra A, any A-module N, and any pointed simplicial set $X_\bullet$, the *(higher) Hochschild chain complex* $CH^{X_\bullet}_\bullet(A,N)$ *of A and N over* $X_\bullet$ as well as the Hochschild cochain complex $CH^\bullet_{X_\bullet}(A,N)$ over $X_\bullet$; see [15, 25]. These Hochschild (co)chain complexes are functorial in all three of their variables A, N and $X_\bullet$.

The analogy with the usual Hochschild complex and its connection to the free loop space is in fact complete, because the Hochschild complex $CH^{X_\bullet}_\bullet(\Omega^\bullet M,\Omega^\bullet M)$ over a simplicial set $X_\bullet$ provides an algebraic model of the differential forms on the mapping space $M^X=\{f:X\to M\}$, where $X=|X_\bullet|$ is the geometric realisation of $X_\bullet$. This is one of the main result of Section 2 of this paper; see Section 2.5. The main tool to prove this result is a machinery of iterated integrals that we develop and use to obtain a quasi-isomorphism $\mathcal{I}t^{X_\bullet} : CH^X_\bullet(\Omega^\bullet M,\Omega^\bullet M) \to \Omega^\bullet(M^X)$, for any k-dimensional simplicial set $X_\bullet$ and k-connected manifold M; see Proposition 2.5.3, and Corollary 2.5.5 for the dual statement. Further, for any simplicial set $X_\bullet$ and (differential graded) commutative algebra A, the Hochschild chain complex $CH^{X_\bullet}_\bullet(A,A)$ has a natural structure of a differential graded commutative algebra given by the shuffle product $\mathrm{sh}_{X_\bullet}$ (Proposition 2.4.2). We show that the iterated integral $\mathcal{I}t^{X_\bullet} : (CH^X_\bullet(\Omega^\bullet M,\Omega^\bullet M),\mathrm{sh}_{X_\bullet}) \to (\Omega^\bullet(M^X),\wedge)$ is an algebra map sending the shuffle product to the wedge product of differential forms on the mapping space; see Proposition 2.4.6.

Two important features of Hochschild (co)chain complexes over simplicial sets are their naturality in the simplicial set $X_\bullet$, and that two simplicial models of quasi-isomorphic spaces have naturally quasi-isomorphic Hochschild (co)chain complexes, see [25]. In particular one (usually) obtains many different models to study $CH^{X_\bullet}_\bullet(A,N)$ for a given space $X=|X_\bullet|$. These facts are used, in Sections 3 and 4, to carry certain geometric and topological constructions over to the Hochschild complexes modeled on compact surfaces Σ^g of genus g.

The collection of compact surfaces of any genus is naturally equipped with a product similar to the loop product of string topology [5], also see [28]. The idea behind this product, that we call the surface product, is shown in the following picture.



(1.1)

In Section 3.1, we describe an explicit simplicial model for the string topology type operation induced by the map

$$\mathrm{Map}(\Sigma^g,M)\times\mathrm{Map}(\Sigma^h,M)\xleftarrow{\rho_{\mathrm{in}}}\mathrm{Map}(\Sigma^g\vee\Sigma^h,M)\xrightarrow{\rho_{\mathrm{out}}}\mathrm{Map}(\Sigma^{g+h},M)$$

coming from the above picture (1.1). More precisely, we obtain a surface product $\uplus : H_\bullet(\mathrm{Map}(\Sigma^g,M))\otimes H_\bullet(\mathrm{Map}(\Sigma^h,M))\to H_{\bullet+\dim(M)}(\mathrm{Map}(\Sigma^{g+h},M))$, which is given by the composition of the umkehr map $(\rho_{\mathrm{in}})_!$ and the map induced by ρ_{out},

$$\begin{aligned}\uplus : H_\bullet(\mathrm{Map}(\Sigma^g,M))\otimes H_\bullet(\mathrm{Map}(\Sigma^h,M))&\\ \xrightarrow{(\rho_{\mathrm{in}})_!} H_\bullet(\mathrm{Map}(\Sigma^g\vee\Sigma^h,M))&\xrightarrow{(\rho_{\mathrm{out}})_*} H_\bullet(\mathrm{Map}(\Sigma^{g+h},M)).\end{aligned}$$

We prove that the surface product makes

$$\Big(\bigoplus_g \mathbb{H}_\bullet(\mathrm{Map}(\Sigma^g,M)),\uplus\Big)=\Big(\bigoplus_g H_{\bullet+\dim(M)}(\mathrm{Map}(\Sigma^g,M)),\uplus\Big)$$

into an associative bigraded [(1)] algebra with $\mathbb{H}_\bullet(\mathrm{Map}(\Sigma^0,M))$ in its center; see Theorem 3.2.2 and Proposition 3.2.5. The restriction of the surface product to genus zero (*i.e.* spheres), $\mathbb{H}_\bullet(\mathrm{Map}(\Sigma^0,M))$, coincides with the Brane topology product defined by Sullivan and Voronov $\mathbb{H}_\bullet(\mathrm{Map}(S^2,M))^{\otimes 2}\to\mathbb{H}_\bullet(\mathrm{Map}(S^2,M))$ see [9, 19]. In fact, in these papers it is shown, that $H_{\bullet+\dim(M)}(M^{S^n})$ is an algebra over $H_\bullet(fD_{n+1})$, the homology of the framed n-disc operad; see also [28] for related algebraic structures.

[(1)] We always use a cohomological grading convention in this paper hence the plus sign in our degree shifting.

In Section 3.3, we apply the machinery of (higher) Hochschild cochain complexes over simplicial sets to give a fully algebraic description of the surface product. In fact, for positive genera, we define an associative cup product

$$\cup : CH^{\bullet}_{\Sigma^g_\bullet}(A,B)\otimes CH^{\bullet}_{\Sigma^h_\bullet}(A,B)\to CH^{\bullet}_{\Sigma^{g+h}_\bullet}(A,B)$$

for the Hochschild cochains over surfaces, where B is a differential graded commutative and unital A-algebra, viewed as a symmetric bimodule; see Definition 3.3.2. The construction of the cup-product is based on the fact that for any pointed simplicial sets $X_\bullet$ and $Y_\bullet$, the multiplication $B\otimes B\xrightarrow{\cdot} B$ induces a cochain map

$$CH^{\bullet}_{\Sigma^g_\bullet}(A,B)\otimes CH^{\bullet}_{\Sigma^h_\bullet}(A,B)\xrightarrow{\vee} CH^{\bullet}_{(\Sigma^g\vee\Sigma^h)_\bullet}(A,B)$$

which can be composed with the pullback $CH^{\bullet}_{(\Sigma^g\vee\Sigma^h)_\bullet}(A,B)\overset{\mathrm{Pinch}^*_{g,h}}{\to} CH^{\bullet}_{\Sigma^{\bullet}_{i+j}}(A,B)$ induced by a simplicial model $\mathrm{Pinch}_{g,h}:\Sigma^{g+h}_\bullet\to\Sigma^g_\bullet\vee\Sigma^h_\bullet$ for the map pinch in Figure (1.1).

However, this product is initially only defined for surfaces of positive genera, and more work is required to include the genus zero case of the 2-sphere in this framework. To this end, we first recall the cup product defined in [15] for genus zero, and then define a left and right action, $\tilde\cup$, of $CH^{\Sigma^0_\bullet}_{\bullet}(A,B)$ on $CH^{\Sigma^g_\bullet}_{\bullet}(A,B)$. Taking advantage of the fact that one can choose different simplicial models for a given space, we show that $\tilde\cup$ is after passing to homology, equivalent to operations similar to the cup-product but defined on different simplicial models for the sphere and surfaces of genus g; see Definition 3.3.13 and Proposition 3.3.14. Putting everything together, we obtain a well-defined cup product $\cup\ :\ \left(\bigoplus_{g\ge0}HH^{\bullet}_{\Sigma^g_\bullet}(A,A)\right)^{\otimes2}\ \to\ \bigoplus_{g\ge0}HH^{\bullet}_{\Sigma^g_\bullet}(A,A)$ for all genera on cohomology; see Theorem 3.3.18. More precisely, we prove that for a differential graded commutative algebra (A,d_A),

i) the cup product makes $\bigoplus_{g\ge0}HH^{\bullet}_{\Sigma^g_\bullet}(A,A)$ into an associative algebra that is bigraded with respect to the total degree grading and the genus of the surfaces and has a unit induced by the unit 1_A of A.
ii) $HH^{\bullet}_{\Sigma^0_\bullet}(A,A)$ lies in the center of $\bigoplus_{g\ge0}HH^{\bullet}_{\Sigma^g_\bullet}(A,B)$.

The cup product is functorial with respect to both arguments A and B (see Proposition 3.3.20).

The connection to topology is precise and the cup-product models the surface product. We prove in Theorem 3.4.2, using rational homotopy techniques as in [11], that for a 2-connected compact manifold M, the (dualized) iterated integral $\mathcal{I}t^{\Sigma^\bullet}\ :\ (\bigoplus_{g\ge0}\mathbb{H}_{-\bullet}(\mathrm{Map}(\Sigma^g,M)),\uplus)\ \to\ (\bigoplus_{g\ge0}HH^{-\bullet}_{\Sigma^g_\bullet}(\Omega,\Omega),\cup)$ is an isomorphism of algebras. As a corollary of this, it follows that the surface product $\uplus$ is homotopy invariant, meaning that if M and N are 2-connected compact manifolds with equal dimensions, and $i:M\to N$ is a homotopy equivalence, then $i_*\ :\ (\bigoplus_{g\ge0}\mathbb{H}_\bullet(\mathrm{Map}(\Sigma^g,M)),\uplus)\to(\bigoplus_{g\ge0}\mathbb{H}_\bullet(\mathrm{Map}(\Sigma^g,N)),\uplus)$ is an isomorphism of algebras.

Section 4 is devoted to the Hochschild homology of (differential graded) symmetric algebras $(S(V),d)$ and a Hochschild-Kostant-Rosenberg type theorem. Recall that classically, when $d=0$, this theorem states that the Hochschild homology $HH_\bullet(S(V),S(V))$, thought of as the Hochschild homology of functions on the dual space V^*, can be identified with