ISSN 0012-9593

quatrième série - tome 44 fascicule 4 juillet-août 2011

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 44, 2011, p. 607 à 629

EXPLICIT COGENERATORS FOR THE HOMOTOPY CATEGORY OF PROJECTIVE MODULES OVER A RING

BY AMNON NEEMAN

ABSTRACT. – Let R be a ring. In two previous articles [12, 14] we studied the homotopy category $\mathbf{K}(R\text{-Proj})$ of projective R-modules. We produced a set of generators for this category, proved that the category is $\aleph_1$-compactly generated for any ring R, and showed that it need not always be compactly generated, but is for sufficiently nice R. We furthermore analyzed the inclusion $j_! : \mathbf{K}(R\text{-Proj}) \longrightarrow \mathbf{K}(R\text{-Flat})$ and the orthogonal subcategory $\mathcal{S} = \mathbf{K}(R\text{-Proj})^\perp$. And we even showed that the inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{-Flat})$ has a right adjoint; this forces some natural map to be an equivalence $\mathbf{K}(R\text{-Proj}) \longrightarrow \mathcal{S}^\perp$.

In this article we produce a set of cogenerators for $\mathbf{K}(R\text{-Proj})$. More accurately, this set of cogenerators naturally lies in the equivalent $\mathcal{S}^\perp \cong \mathbf{K}(R\text{-Proj})$; it can be used to give yet another proof of the fact that the inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{-Flat})$ has a right adjoint. But by now several proofs of this fact already exist.

RÉSUMÉ. – Soit R un anneau. Dans deux articles antérieurs [12, 14], on a étudié la catégorie d'homotopie $\mathbf{K}(R\text{-Proj})$ des R-modules projectifs. On a construit un ensemble de générateurs pour cette catégorie et on a démontré que la catégorie est compactement générée de niveau $\aleph_1$ pour chaque anneau R, mais qu'elle n'est pas toujours compactement générée. Toutefois, pour R un anneau suffisamment raisonnable, la catégorie $\mathbf{K}(R\text{-Proj})$ est compactement générée. On a étudié l'inclusion $j_! : \mathbf{K}(R\text{-Proj}) \longrightarrow \mathbf{K}(R\text{-Flat})$ et la sous-catégorie orthogonale $\mathcal{S} = \mathbf{K}(R\text{-Proj})^\perp$. On a même montré que l'inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{-Flat})$ admet un adjoint à droite ; il s'ensuit qu'une certaine application naturelle $\mathbf{K}(R\text{-Proj}) \longrightarrow \mathcal{S}^\perp$ est une équivalence.

Dans le présent article, on produit un ensemble de cogénérateurs pour $\mathbf{K}(R\text{-Proj})$. Plus précisément, cet ensemble de cogénérateurs appartient naturellement à la catégorie équivalente $\mathcal{S}^\perp \cong \mathbf{K}(R\text{-Proj})$; on peut l'utiliser pour obtenir une nouvelle démonstration du fait que l'inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{-Flat})$ admet un adjoint à droite. Mais il y a déjà plusieurs autres démonstrations de ce fait.

The research was partly supported by the Australian Research Council.

0. Introduction

Let $\mathscr{T}$ be a triangulated category with products. A subcategory $\mathscr{S} \subset \mathscr{T}$ is called *colocalizing* if it is triangulated and closed under products. Given any class of objects $T \subset \mathscr{T}$, the smallest colocalizing subcategory containing T will be denoted $\mathrm{Coloc}(T)$, and referred to as the colocalizing subcategory cogenerated by T. If $\mathrm{Coloc}(T) = \mathscr{T}$, then we say that T cogenerates $\mathscr{T}$. If $\mathscr{T}$ has coproducts (that is, $\mathscr{T}^{\mathrm{op}}$ has products) then we can dualize: a class of objects $T \subset \mathscr{T}$ is said to generate $\mathscr{T}$ if it cogenerates $\mathscr{T}^{\mathrm{op}}$.

For various reasons it is interesting to try to find *sets* of objects (as opposed to classes) $T \subset \mathscr{T}$ which cogenerate $\mathscr{T}$. One reason is that all known proofs of the Brown representability theorem depend on producing sets of generators or, dually, of cogenerators. Embarrassingly, the situation is not symmetric. We often know how to produce a set of generators for some category $\mathscr{T}$, without having the foggiest clue whether the category has a set of cogenerators. Thus Brown's original proof of his representability theorem in [2], which looked at the case where $\mathscr{T}$ was the homotopy category of spectra, depended on the fact that the spheres are compact generators. It took 36 years before anyone noticed that this category has a sufficiently nice set of cogenerators so that Brown representability also holds for the dual; see [10].

By now it is known how to produce a set of cogenerators in any compactly generated triangulated category; there are two somewhat different treatments of the same set of cogenerators in Krause [6] and in [11], together with proofs that its existence implies Brown representability for the dual. Assuming the category $\mathscr{T}$ has a Rosický functor then there is a completely different construction of a set of cogenerators in [13], and, once again, an argument showing that these cogenerators are nice enough to force Brown representability to hold in the dual. But it is not clear how useful this observation is; ever since Rosický retracted his result of [15], we know of relatively few examples where we can directly produce Rosický functors. Unfortunately this is all we know at present, there is no known general method to produce cogenerators in categories that happen not to be compactly generated. Hence the interest of this article: we give a construction of cogenerators in categories known to be well generated but not compactly generated.

Let R be an associative ring with a unit. Let $\mathbf{K}(R\text{–Proj})$ be the homotopy category of chain complexes of projective R-modules. The main result of this article amounts to producing an explicit set of cogenerators. But there is a technical wrinkle that we should now explain: the actual cogenerators we choose naturally lie not in $\mathbf{K}(R\text{–Proj})$ but in an equivalent category that we will often denote by $\mathscr{S}^{\perp}$. It is now time to elaborate.

In [12, Proposition 8.1] we proved that the natural inclusion $j_! : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$ has a right adjoint $j^* : \mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Proj})$. In [14, Theorem 3.1] we showed that the functor j^* has a right adjoint $j_* : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$. The functor $j_!$ is obviously fully faithful and it follows formally, from general nonsense about triangulated categories, that so is the functor j_*. Our cogenerators can naturally be described as objects in the essential image of j_*.

Let us rephrase this a little. We have a fully faithful functor $j_! : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$, namely the obvious embedding, and it possesses a right adjoint j^*. Define $\mathscr{S} = \mathbf{K}(R\text{–Proj})^{\perp}$ to be the full subcategory of $\mathbf{K}(R\text{–Flat})$ whose objects are

$$\mathrm{Ob}(\mathscr{S}) \quad = \quad \{y \in \mathbf{K}(R\text{–Flat}) \mid \mathrm{Hom}(j_! x, y) = 0 \ \ \forall x \in \mathbf{K}(R\text{–Proj})\} \,.$$

Then the essential image of the functor $j_* : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$ is precisely the subcategory $\mathscr{S}^\perp \subset \mathbf{K}(R\text{–Flat})$. Our main theorem proves that a certain explicit set of objects in $\mathscr{S}^\perp$ cogenerates. In order to write down these explicit cogenerators we remind the reader:

REMINDER 0.1. – The natural inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$ has a right adjoint $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$; the first proof appeared in [14, Theorem 3.2], but since then there have been other proofs of more general results: see Bravo, Enochs, Iacob, Jenda and Rada [1, Theorem 3.5 and the third Example in §4], Krause [8, Corollary 3.4], and Saorín and Šťovíček [16, Proposition 4.12].

The following statement summarizes the main results of the paper:

THEOREM 0.2. – *Let the notation be as above. In particular, let* $\mathscr{S} = \mathbf{K}(R\text{–Proj})^\perp$ *be the category of all* $\mathbf{K}(R\text{–Proj})$*-local objects in* $\mathbf{K}(R\text{–Flat})$*. Then the objects of* $\mathscr{S}^\perp = \{\mathbf{K}(R\text{–Proj})^\perp\}^\perp$ *are cogenerated by the chain complexes* $J\big(\mathscr{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big) \in \mathbf{K}(R\text{–Flat})$*, where*

1. *The functor J is the right adjoint to the inclusion* $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$.
2. *I runs over the bounded below chain complexes of injective right R-modules, which satisfy the following two conditions:*
   (a) *All but finitely many of the groups $H^i(I)$ vanish.*
   (b) *For all i, $H^i(I)$ is isomorphic to a subquotient of a finitely generated, projective right R-module.*

The proof may be found in Theorem 4.7.

REMARK 0.3. – In the process of proving Theorem 4.7 we discover that we also give yet another proof of the existence of a right adjoint to the natural inclusion $i_* : \mathscr{S} \longrightarrow \mathbf{K}(R\text{–Flat})$. More explicitly the argument goes as follows: it is easy to show that the objects $J\big(\mathscr{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$ all lie in $\mathscr{S}^\perp$; see Remark 2.7. One immediately deduces the inclusion $\mathrm{Coloc}(S) \subset \mathscr{S}^\perp$. Given an object $y \in \mathbf{K}(R\text{–Flat})$ we will show how to construct a triangle

$$s \longrightarrow y \longrightarrow t \longrightarrow \Sigma s$$

with $s \in \mathscr{S}$ and $t \in \mathrm{Coloc}(S)$; this will automatically prove both the existence of the right adjoint to the functor i_* and the fact that $\mathrm{Coloc}(S) = \mathscr{S}^\perp$.

REMARK 0.4. – When the ring R is right coherent, we find that the proof of Theorem 4.7 can be done without making use of the functor J. The argument of this paper simplifies substantially when R is right coherent. Part of the reason is that, when R is right coherent, the category $K(R\text{–Proj})$ is compactly generated, and for compactly generated categories there is a standard way to construct cogenerators. The standard cogenerators are the Brown-Comenetz duals of the compact objects; in §5 we will see that, for right coherent R, the set of cogenerators we give in Theorem 0.2 includes the standard ones.

The remarkable feature of our construction of cogenerators is its generality. For general R the category $K(R\text{–Proj})$ is only well generated and not compactly generated, and there is no known procedure to construct cogenerators. It is known that the recipe that works for compactly generated categories *does not* generalize; this can be restated as saying that a certain abelian category does not have enough injectives, see [11, Appendix D.2]. What we produce here amounts to the first known example of cogenerators in a non-compactly-generated but well generated triangulated category.

REMARK 0.5. – In the special case where R is commutative, noetherian and of finite Krull dimension there is already a discussion of $\mathscr{S}^\perp$ in the literature; see Enochs and Garcia [3, Theorem 4.6]. We should also mention the growing literature on related topics: see Bravo, Enochs, Iacob, Jenda and Rada [1], Jørgensen [5], Iyengar and Krause [4], Krause [7, 8], Murfet [9], and Saorín and Šťovíček [16].

Acknowledgements

I would like to thank the referee for helpful suggestions on how to improve an earlier version.

1. Tensor-phantom maps

This section is devoted to technical preliminaries. The highlights are Definitions 1.1 and 1.3, as well as Lemma 1.9. The reader wishing to form an overall impression of the proof might wish to read just the two definitions and the statement of the lemma, and then proceed to §2. The current section contains the hard work in the article, but the reader might prefer to first have an idea why we bother.

DEFINITION 1.1. – *A* test-complex *I is a bounded below chain complex of injective right R-modules, with $H^i(I) = 0$ for all but finitely many $i \in \mathbb{Z}$. For those $i \in \mathbb{Z}$ for which $H^i(I) \neq 0$, we insist that $H^i(I)$ must be isomorphic to a subquotient of a finitely generated, projective right R-module.*

REMARK 1.2. – The definition is intended to ensure that, up to homotopy equivalence, there is only a set of test-complexes. Up to isomorphism, the collection of finitely generated, projective right R-modules forms a set. Hence so do all their subquotients. Therefore the triangulated subcategory $\mathscr{R}$, that these subquotients generate in $\mathbf{D}^b(R\text{–Mod})$, is essentially small. The test-complexes are injective resolutions of some of the objects in $\mathscr{R}$; there is only a set of them, up to homotopy equivalence.

DEFINITION 1.3. – *Let Y and Z be objects in $\mathbf{K}(R\text{–Flat})$. A morphism $f : Y \longrightarrow Z$ is called* tensor-phantom *if, for every test-complex I as in Definition 1.1, the map*

$$I \otimes_R Y \xrightarrow{1\otimes f} I \otimes_R Z$$

vanishes in cohomology. That is, the induced maps $H^i(I \otimes_R Y) \longrightarrow H^i(I \otimes_R Z)$ all vanish.

REMARK 1.4. – The tensor-phantom maps form an ideal, in the category $\mathbf{K}(R\text{–Flat})$. We remind the reader: this means

(i) If $g, g' : Y \longrightarrow Z$ are two tensor-phantom maps, then $g + g'$ is also tensor-phantom.
(ii) If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow Z'$ are maps of chain complexes, and if g is tensor-phantom, then $gf : X \longrightarrow Z$ and $hg : Y \longrightarrow Z'$ are also tensor-phantom.