ISSN 0012-9593

quatrième série - tome 45 fascicule 6 novembre-décembre 2012

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Thomas GAUTHIER

Strong bifurcation loci of full Hausdorff dimension

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 45, 2012, p. 947 à 984

STRONG BIFURCATION LOCI OF FULL HAUSDORFF DIMENSION

BY Thomas GAUTHIER

ABSTRACT. – In the moduli space $\mathcal{M}_d$ of degree d rational maps, the bifurcation locus is the support of a closed $(1,1)$ positive current T_{bif} which is called the bifurcation current. This current gives rise to a measure $\mu_{\text{bif}} := (T_{\text{bif}})^{2d-2}$ whose support is the seat of strong bifurcations. Our main result says that $\text{supp}(\mu_{\text{bif}})$ has maximal Hausdorff dimension $2(2d-2)$. As a consequence, the set of degree d rational maps having $(2d-2)$ distinct neutral cycles is dense in a set of full Hausdorff dimension.

RÉSUMÉ. – Dans l'espace des modules $\mathcal{M}_d$ des fractions rationnelles de degré d, le lieu de bifurcation est le support d'un $(1,1)$-courant positif fermé T_{bif} qui est appelé courant de bifurcation. Ce courant induit une mesure $\mu_{\text{bif}} := (T_{\text{bif}})^{2d-2}$ dont le support est le siège de bifurcations maximales. Notre principal résultat stipule que $\text{supp}(\mu_{\text{bif}})$ est de dimension de Hausdorff maximale $2(2d-2)$. Par conséquent, l'ensemble des fractions rationnelles de degré d possédant $(2d-2)$ cycles neutres distincts est dense dans un ensemble de dimension de Hausdorff totale.

1. Introduction

The boundary of the Mandelbrot set has Hausdorff dimension 2. This fundamental result is the main Theorem of Shishikura's work [29]. Tan Lei has generalized this by showing that the boundary of the connectedness locus of polynomial families of any degree has maximal Hausdorff dimension. Tan Lei has also shown that the bifurcation locus in any non-stable holomorphic family of rational maps is of full dimension (see [34]). McMullen gave another proof of her result in [23]. Our aim here is to show that dynamically relevent, but a priori much smaller, subsets of the bifurcation locus have maximal Hausdorff dimension in the space Rat_d of all degree d rational maps.

We can define a *bifurcation current* on Rat_d by setting $T_{\text{bif}} := dd^cL$, where $L(f)$ is the Lyapounov exponent of f with respect to its maximal entropy measure. DeMarco has shown in [13] and [14] that the support of T_{bif} is precisely the bifurcation locus. This current and its powers T_{bif}^k ($k \leq 2d-2$) have been used in several recent works for studying the geometry of the bifurcation locus (see [3, 4, 5, 11, 15, 17, 18]). Moebius transformations act by conjugacy on Rat_d and the quotient space is an orbifold known as the *moduli space* $\mathcal{M}_d$ of degree d

rational maps. Giovanni Bassanelli and François Berteloot [3] introduced a measure μ_{bif} on this moduli space, which may be obtained by pushing T_{bif}^{2d-2} forward. We will call *strong bifurcation locus* the support of this measure μ_{bif}. This set can be interpreted as a set on which bifurcations are maximal. Our main result is the following:

THEOREM 1.1. – *The support of the bifurcation measure μ_{bif} of the moduli space $\mathcal{M}_d$ of degree d rational maps is homogeneous and has maximal Hausdorff dimension, i.e.,*

$$\dim_H(\mathrm{supp}(\mu_{\mathrm{bif}}) \cap \Omega) = 2(2d-2)$$

for any open set $\Omega \subset \mathcal{M}_d$ such that $\mathrm{supp}(\mu_{\mathrm{bif}}) \cap \Omega \neq \varnothing$.

Let us mention that this implies that the conjugacy classes of rational maps having $2d-2$ distinct neutral cycles are dense in a homogeneous set of full Hausdorff dimension in $\mathcal{M}_d$ (see Main Theorem of [11]).

As we shall now explain, this will be obtained by using Misiurewicz rational maps properties and bifurcation currents techniques. A rational map f is *k-Misiurewicz* if its Julia set contains exactly k critical points counted with multiplicity, if f has no parabolic cycle and if the ω-limit set of any critical point in its Julia set does not meet the critical set. A classical result of Mañé states that the k critical points of f which are in $\mathcal{J}_f$ eventually fall under iteration in a compact f-hyperbolic set E_0, which means that f is uniformly expanding on E_0, and that the $2d-2-k$ remaining critical points are in attracting basins of f (see Section 2.2).

The first result we need to establish is the following transversality theorem:

THEOREM 1.2 (Weak transversality). – *Let $(f_\lambda)_{\lambda\in\mathbb{B}(0,r)}$ be a holomorphic family of degree d rational maps parametrized by a ball $\mathbb{B}(0,r) \subset \mathbb{C}^{2d-2}$ with $2d-2$ marked critical points. Let f_0 be k-Misiurewicz but not a flexible Lattès map. Let us denote by E_0 the compact f_0-hyperbolic set such that $f_0^{k_0}(c_1(0)), \ldots, f_0^{k_0}(c_k(0)) \in E_0$. Denote also by h the dynamical holomorphic motion of E_0. If the set $\{\lambda \mid \exists m \in \mathrm{Aut}(\mathbb{P}^1), f_\lambda \circ m = m \circ f_{\lambda_0}\}$ is discrete for any $\lambda_0 \in \mathbb{B}(0,r)$, then*

$$\mathrm{codim}\,\{\lambda \in \mathbb{B}(0,r) \mid f_\lambda^{k_0}(c_j(\lambda)) = h_\lambda(f_0^{k_0}(c_j(0))), 1 \leq j \leq k\} = k.$$

The establishment of Theorem 1.2 is the subject of Section 3. Let us mention that a stronger transversality result has been proved by van Strien in [33] in the case of $(2d-2)$-Misiurewicz maps with a trivial stabilizer for the action by conjugation of the group $\mathrm{Aut}(\mathbb{P}^1)$ on Rat_d, and by Buff and Epstein in [11] in the case of strictly postcritically finite rational maps. We want here to give a weaker result which is easier to prove and sufficient for our purpose. In their work [11], Buff and Epstein established their tranversality Theorem using quadratic differential techniques. As these tools are not well adapted when the critical orbits are infinite, we have instead followed van Strien's and Aspenberg's ideas (see [33] and [1]) of using quasiconformal maps. See also [27] for a transversality result concerning quadratic semihyperbolic polynomials.

Section 4 is devoted to local dimension estimates. To achieve our goal we have to prove that the set of $(2d-2)$-Misiurewicz maps has maximal Hausdorff dimension in Rat_d. Like in Shishikura's or Tan Lei's work, this basically requires to "copy" big hyperbolic sets in the parameter space. For this, using the transversality Theorem 1.2, we construct a transfer map from the dynamical plane to the parameter space which enjoys good regularity properties.

When $k = 1$, our proof here is actually slightly simpler than the classical ones. Indeed, in his original proof, Shishikura builds two successive holomorphic motions, where we only use one such motion. Using the work of Shishikura [29] on parabolic implosion and Theorem 1.3, and using again the transversality Theorem 1.2, we deduce that the set $\mathfrak{M}_k$ is homogeneous and has maximal Hausdorff dimension $2(2d+1)$. The main result of Section 4 is the following:

THEOREM 1.3. – *Let $1 \leq k \leq 2d-2$. Denote by $\mathfrak{M}_k$ the set of all k-Misiurewicz degree d rational maps with simple critical points which are not flexible Lattès maps. Then for any $f \in \mathfrak{M}_k$ and any neighborhood $V_0 \subset \mathrm{Rat}_d$ of f one has:*

$$\dim_H(\mathfrak{M}_k \cap V_0) \geq 2(2d+1-k) + k \dim_{\mathrm{hyp}}(f).$$

For our purpose we have to show that k-Misiurewicz maps belong to the support of T^k_{bif}. To achieve this goal, we give a criterion for a rational map with k critical points eventually falling under iteration in a compact hyperbolic set to belong to the support of T^k_{bif}. The condition is exactly the one that appears in the statement of the Transversality Theorem 1.2. Our approach is inspired by the work of Buff and Epstein [11] who proved that strictly postcritically finite maps, which are $(2d-2)$-Misiurewicz, belong to the support of T^{2d-2}_{bif}. The main idea of the proof of this criterion is a dynamical renormalization process which enlightens a similarity phenomenon between parameter and dynamical spaces at maps with critical points eventually falling into a hyperbolic set. To implement this process we need to linearize the map along a repelling orbit. In the geometrically finite case this is just the linearization along repelling cycles. This technical result, which may be of independent interest, is established in Section 5. The renormalization is then performed in Section 6. In the same section, we also prove that the k^{th}-self-intersection of the bifurcation current detects the activity of at least k distinct critical points (see Theorem 6.1). Combining the results of Section 6 with Theorem 1.2, we get the following:

THEOREM 1.4. – *Let $1 \leq k \leq 2d-2$ and $f \in \mathrm{Rat}_d$ be k-Misiurewicz but not a flexible Lattès map. Then $f \in \mathrm{supp}(T^k_{\mathrm{bif}}) \setminus \mathrm{supp}(T^{k+1}_{\mathrm{bif}})$.*

In Section 7, we focus on dimension estimates for the bifurcation measure. The estimates established in Section 4 have the following interesting consequence. Due to Theorem 1.4, they directly imply that the set $\mathrm{supp}(T^k_{\mathrm{bif}}) \setminus \mathrm{supp}(T^{k+1}_{\mathrm{bif}})$ has Hausdorff dimension $2(2d+1)$ for $1 \leq k \leq 2d-3$, and that $\mathrm{supp}(T^{2d-2}_{\mathrm{bif}})$ is homogeneous and that the set $\dim_H(\mathrm{supp}(T^{2d-2}_{\mathrm{bif}})) = 2(2d+1)$ (see Theorem 7.1). Notice that it is still unknown wether $\mathrm{supp}(T^k_{\mathrm{bif}})$ is homogeneous or not, when $1 \leq k \leq 2d-3$. Theorem 1.1 then immediately follows. Recall that, if μ is a Radon measure on a metric space, the *upper pointwise dimension* of μ at x_0 is

$$\overline{\dim}_\mu(x_0) := \limsup_{r \to 0} \frac{\log \mu(\mathbb{B}(x_0, r))}{\log r}.$$

In the same section, we also establish a lower bound for the upper pointwise dimension of the bifurcation measure at $(2d-2)$-Misiurewicz parameters, which rely on the renormalization process performed in Section 6. In particular, we get the following:

THEOREM 1.5. – *There exists a dense subset* $\mathfrak{M}$ *of* $\text{supp}(\mu_{\text{bif}})$ *which is homogeneous and has maximal Hausdorff dimension* $2(2d-2)$ *such that* $\overline{\dim}_{\mu_{\text{bif}}}[f] > 0$ *for any* $[f] \in \mathfrak{M}$.

Let us mention that all these results have their counterpart in polynomial families. We develop the description of the above results in the context of polynomial families in Section 8. For instance, the equivalent of Theorem 1.1 might be stated as follows:

THEOREM 1.6. – *In the moduli space* $\mathcal{P}_d$ *of degree d polynomials, the Shilov boundary of the connectedness locus is homogeneous and has maximal Hausdorff dimension* $2(d-1)$.

This improves Tan Lei's result. We also give a very easy proof of Theorem 1.2 in the case of a degree d polynomial with k critical points which are preperiodic to repelling cycles (see Lemma 8.2).

1.1. Notation

To end the introduction, let us give some notation:

- $\mathbb{P}^1$ is the Riemann sphere, $\mathbb{D}$ is the unit disc of $\mathbb{C}$ and $\mathbb{D}(0,r)$ is the disc of $\mathbb{C}$ of radius r centered at 0. Leb denotes the Lebesgue measure on $\mathbb{P}^1$.
- $\|\cdot\|$ is a norm on $\mathbb{C}^n$ and $\mathbb{B}(a,r)$ is the ball of $\mathbb{C}^n$ of radius r centered at $a \in \mathbb{C}^n$.
- $\mathcal{J}_f$ is the Julia set and $\mathcal{F}_f$ is the Fatou set of $f \in \text{Rat}_d$.
- The group $\text{Aut}(\mathbb{P}^1)$ is the group of all Moebius transformations. The quotient space $\mathcal{M}_d := \text{Rat}_d/\text{Aut}(\mathbb{P}^1)$ is the *moduli space* of degree d rational maps and $\Pi : \text{Rat}_d \longrightarrow \mathcal{M}_d$ is the quotient map. The space $\mathcal{M}_d$ has a canonical structure of affine variety of dimension $2d-2$ (see [32] pages 174–179).
- Finally, $\dim_H$ is the Hausdorff dimension in any metric space.

1.2. Acknowledgment

The author would very much like to thank François Berteloot for his help and encouragement for writing this article. The author also wants to thank Romain Dujardin for his helpful comments and for suggesting Lemma 8.2 and the formulation of Theorem 6.2. Finally, we would like to thank the Referees for their very useful suggestions, which have greatly improved the readability of this work.

2. The framework

2.1. Hyperbolic sets

DEFINITION 2.1. – *Let* $f \in \text{Rat}_d$ *and* $E \subset \mathbb{P}^1$ *be a compact* f*-invariant set, i.e., such that* $f(E) \subset E$. *We say that* E *is* f-hyperbolic *if one of the following equivalent conditions is satisfied:*

1. *there exist constants* $C > 0$ *and* $\alpha > 1$ *such that* $|(f^n)'(z)| \geq C\alpha^n$ *for all* $z \in E$ *and all* $n \geq 0$,
2. *for some appropriate metric on* $\mathbb{P}^1$, *there exists* $K > 1$ *such that* $|f'(z)| \geq K$ *for all* $z \in E$. *One says that* K *is the* hyperbolicity constant *of* E.