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ON STANDARD NORM VARIETIES

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ABSTRACT. – Let p be a prime integer and F a field of characteristic 0. Let X be the *norm variety* of a symbol in the Galois cohomology group $H^{n+1}(F,\mu_p^{\otimes n})$ (for some $n \geq 1$), constructed in the proof of the Bloch-Kato conjecture. The main result of the paper affirms that the function field $F(X)$ has the following property: for any equidimensional variety Y, the change of field homomorphism $\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)})$ of Chow groups with coefficients in integers localized at p is surjective in codimensions $< (\dim X)/(p-1)$. One of the main ingredients of the proof is a computation of Chow groups of a (generalized) Rost motive (a variant of the main result not relying on this is given in the appendix). Another important ingredient is *A-triviality* of X, the property saying that the degree homomorphism on $\mathrm{CH}_0(X_L)$ is injective for any field extension L/F with $X(L) \neq \varnothing$. The proof involves the theory of rational correspondences reviewed in the appendix.

RÉSUMÉ. – Pour un nombre premier p et un corps F de caractéristique 0, soit X la *variété de norme* d'un symbole dans le groupe de cohomologie galoisienne $H^{n+1}(F,\mu_p^{\otimes n})$ (avec $n \geq 1$) construite au cours de la démonstration de la conjecture de Bloch-Kato. Le résultat principal de cet article affirme que le corps des fonctions $F(X)$ a la propriété suivante : pour toute variété équidimensionnelle Y, l'homomorphisme de changement de corps $\mathrm{CH}(Y) \to \mathrm{CH}(Y_{F(X)})$ de groupes de Chow à coefficients entiers localisés en p est surjectif en codimension $< (\dim X)/(p-1)$. Une des composantes principales de la preuve est le calcul de groupes de Chow du motif de Rost généralisé (un variant du résultat principal indépendant de ceci est proposé dans l'appendice). Un autre ingrédient important est la *A-trivialité* de X, la propriété qui dit que pour toute extension de corps L/F avec $X(L) \neq \varnothing$, l'homomorphisme de degré pour $\mathrm{CH}_0(X_L)$ est injectif. La preuve fait apparaître la théorie de correspondances rationnelles revue dans l'appendice.

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1. Introduction

Let n be a positive integer and p a prime integer. A smooth complete geometrically irreducible variety X over a field F of characteristic 0 is a *p-generic splitting variety* for a symbol $s \in H^{n+1}(F;\mu_p^{\otimes n})$ if s vanishes over a field extension K/F if and only if X over K has a closed point of degree prime to p. A *norm variety* of s is a p-generic splitting variety of the smallest dimension $p^n - 1$. Norm varieties played an important role in the proof of the Bloch-Kato conjecture (see [35]).

Let Y be a smooth variety over F. Write $\mathrm{CH}^i(Y)$ for the Chow group with coefficients in the integers localized at p and $\widetilde{\mathrm{CH}}^i(Y)$ for the factor group of the Chow group $\mathrm{CH}^i(Y)$ modulo p-torsion elements and $p\,\mathrm{CH}^i(Y)$. In [36, Theorem 1.3], K. Zainoulline proved, using the Landweber-Novikov operations in algebraic cobordism theory, that every Y and every norm variety X of s enjoy the following property: if $i < (p^n - 1)/(p - 1)$, every class α in $\widetilde{\mathrm{CH}}^i(Y_{\bar F})$, where $\bar F$ is an algebraic closure of F, such that $\alpha_{\bar F(X)}$ is $F(X)$-rational, is F-rational itself, i.e., α belongs to the image of the map $\widetilde{\mathrm{CH}}^i(Y) \to \widetilde{\mathrm{CH}}^i(Y_{\bar F})$. This statement is in the spirit of the Main Tool Lemma of A. Vishik [32].

In the present paper we improve this result by showing that every cycle in the Chow group $\mathrm{CH}^i(Y_{F(X)})$ is already defined over F, i.e, it comes from $\mathrm{CH}^i(Y)$. More precisely, we prove the following theorem (see Theorem 4.3 for a stronger statement and the proof):

THEOREM 1.1. – *Let F be a field of characteristic* 0 *and let X be an A-trivial p-generic splitting variety of a symbol in $H^{n+1}(F,\mu_p^{\otimes n})$. Then the change of field homomorphism $\mathrm{CH}^i(Y) \to \mathrm{CH}^i(Y_{F(X)})$ is surjective if $i < (p^n - 1)/(p - 1)$ for any equidimensional (not necessarily smooth) variety Y over F. Moreover, the bound $(p^n - 1)/(p - 1)$ is sharp.*

The A-triviality property for X means that the degree map $\deg : \mathrm{CH}_0(X_K) \to \mathbb{Z}_{(p)}$ is an isomorphism (i.e., the kernel $A(X_K)$ of the degree map is trivial) for any field extension K/F such that X has a point over K. (We believe that the A-triviality condition should be also imposed in the statement of [36, Theorem 1.3].) Our proof of Theorem 1.1 is “elementary" in the sense that it does not use the algebraic cobordism theory. It is based on computation of Chow groups of the corresponding Rost motive, an approach applied by A. Vishik in [33, Remark on Page 665] in order to obtain the conclusion of Theorem 1.1 for Pfister quadrics (with $p = 2$).

In Section 5 we prove that the standard norm varieties (corresponding to nontrivial symbols) constructed in [30] are A-trivial, so that Theorem 1.1 can be applied to such varieties. with $p = 2$, In fact, we prove more (see Theorem 5.8 for a more explicit statement and the proof):

THEOREM 1.2. – *Let X be a standard norm variety of a nontrivial symbol over a field F of characteristic* 0*. Then for any field extension K/F, the degree map* $\deg : \mathrm{CH}_0(X_K) \to \mathbb{Z}_{(p)}$ *is injective.*

In the proof we use the theory of rational correspondences developed by M. Rost (unpublished) and B. Kahn/R. Sujatha in [13]. We review this theory in Appendix RC. Another ingredient of the proof, a computation of Chow groups of Rost motives, is presented in

Appendix RM. A variant of the main theorem valid in any characteristic $\neq p$ and involving the Steenrod operations is given in Appendix SC.

In Sections 3 and 4 we develop a theory of (abstract) Rost motives.

We use the following notation and conventions. The base field F is of arbitrary characteristic if not specified otherwise (it is of characteristic $\neq p$, p a fixed prime, most of the time, and of characteristic 0 in several places). An *F-variety* over a field F is a separated scheme of finite type over F.

We fix a commutative unital ring Λ and write $\mathrm{CH} = \mathrm{CH}_*$ for the Chow group with coefficients in Λ. For any integer i and equidimensional F-variety Y, we write $\mathrm{CH}^i(Y)$ for the Chow group $\mathrm{CH}_{\dim Y - i}(Y)$.

Many events in the paper happen in the *category of Chow motives* (with coefficients in Λ, see [7]). A *Chow motive* is a pair (X, ρ), where X is a smooth complete variety over F and ρ is a projector (idempotent) in the endomorphism ring of the motive $M(X)$ of X. We say that a motive M *lives on* X, if $M \simeq (X, \rho)$ for some ρ as above.

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2. A-trivial varieties

Let X be a smooth complete irreducible F-variety. Let d be its dimension, and let ρ be a fixed element of the Chow group $\mathrm{CH}^d(X \times X)$ (considered as a correspondence $X \rightsquigarrow X$).

LEMMA 2.1. – *The following two conditions on ρ are equivalent:*

1. *for any F-variety Y, the image of any $\alpha \in \mathrm{CH}(X \times Y)$ under the pull-back to $\mathrm{CH}(Y_{F(X)})$ coincides with the image of $\alpha \circ \rho$;*
2. *$\rho_*[\xi] = [\xi]$, where ξ is the generic point of X and $[\xi]$ is its class in the Chow group $\mathrm{CH}_0(X_{F(X)})$.*

Proof. – (2) $\Rightarrow$ (1) The image of α is equal to $\alpha_*[\xi]$. In particular, the image of $\alpha \circ \rho$ is equal to $(\alpha \circ \rho)_*[\xi] = \alpha_*(\rho_*[\xi]) = \alpha_*[\xi]$ if $\rho_*[\xi] = [\xi]$.

(1) $\Rightarrow$ (2) Apply (1) to $Y = X$ and the class of the diagonal of X in place of α. □

COROLLARY 2.2. – *If ρ satisfies the conditions of Lemma 2.1, then for any F-variety Y the pull-back homomorphism $\mathrm{CH}(X \times Y) \circ \rho \to \mathrm{CH}(Y_{F(X)})$ is surjective.* □

DEFINITION 2.3. – A smooth complete F-variety X is *A-trivial*, if for any field extension L/F with $X(L) \neq \varnothing$, the degree homomorphism $\deg : \mathrm{CH}_0(X_L) \to \Lambda$ is an isomorphism.

REMARK 2.4. – The notion of A-triviality depends on Λ. A variety A-trivial for $\Lambda = \mathbb{Z}$ is A-trivial for any Λ. If $\Lambda \neq 0$, any A-trivial variety is geometrically irreducible.

EXAMPLE 2.5. – Any projective homogeneous variety X under an action of a semisimple affine algebraic group is A-trivial. Indeed, if $X(L) \neq \varnothing$, the variety X_L is rational and therefore $\deg : \mathrm{CH}_0(X_L) \to \Lambda$ is an isomorphism by Corollary RC.13.

Multiplicity $\operatorname{mult}\rho$ of ρ is the element of Λ such that the push-forward of ρ with respect to the first projection $X \times X \to X$ is equal to $(\operatorname{mult}\rho)\cdot[X]$.

LEMMA 2.6. – *Assuming that X is A-trivial, ρ satisfies conditions of Lemma* 2.1 *if and only if* $\operatorname{mult}\rho = 1$.

Proof. – Since X is A-trivial, the 0-cycle classes $\rho_*[\xi], [\xi] \in \mathrm{CH}_0(X_{F(X)})$ coincide if and only if their degrees coincide. It remains to notice that $\deg[\xi] = 1$ and $\deg \rho_*[\xi] = \operatorname{mult}\rho$. □

A trivial example of ρ satisfying the conditions of Lemma 2.1 is given by the class of the diagonal of X. Here is one more example:

EXAMPLE 2.7. – If X is a projective homogeneous variety under an action of a semisimple affine algebraic group and $\rho \in \mathrm{CH}^d(X \times X)$ is a projector such that the summand (X, ρ) of the Chow motive of X is *upper* in the sense of [17, Definition 2.10], then $\operatorname{mult}\rho = 1$ and therefore ρ satisfies conditions of Lemma 2.1 by Lemma 2.6 (X is A-trivial by Example 2.5).

PROPOSITION 2.8. – *Assume that ρ satisfies conditions of Lemma* 2.1 *(the assumption is satisfied, for instance, if* $\operatorname{mult}\rho = 1$ *and X is A-trivial). Also assume that ρ is a projector. Given an equidimensional F-variety Y and an integer m such that for any i and any point $y \in Y$ of codimension i the change of field homomorphism*

$$\rho^* \mathrm{CH}^{m-i}(X) \to \rho^* \mathrm{CH}^{m-i}(X_{F(y)})$$

is surjective, the change of field homomorphism

$$\mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y_{F(X)})$$

is also surjective.

Proof. – Since $\rho^*(x) \times y = (x \times y) \circ \rho$ for any $x \in \mathrm{CH}(X)$, F-variety Y and $y \in \mathrm{CH}(Y)$ (where the composition of correspondences is taken in the sense of [3], see also [7, §62]), the external product homomorphism $\mathrm{CH}(X) \otimes \mathrm{CH}(Y) \to \mathrm{CH}(X \times Y)$ maps $\big(\rho^* \mathrm{CH}(X)\big) \otimes \mathrm{CH}(Y)$ to $\mathrm{CH}(X \times Y) \circ \rho$.

Let us check that in our situation the homomorphism

$$\bigoplus_i \big(\rho^* \mathrm{CH}^i(X)\big) \otimes_\Lambda \mathrm{CH}^{m-i}(Y) \to \mathrm{CH}^m(X \times Y) \circ \rho$$

is surjective.

Checking this, we may assume that Y is integral and proceed by induction on $\dim Y$ using the exact sequence

$$\bigoplus_{Y'} \mathrm{CH}^{m-1}(X \times Y') \circ \rho \to \mathrm{CH}^m(X \times Y) \circ \rho \to \rho^* \mathrm{CH}^m(X_{F(Y)}),$$

where the direct sum is taken over all integral subvarieties $Y' \subset Y$ of codimension 1. The sequence is exact because the sequence

$$\bigoplus_{Y'} \mathrm{CH}^{m-1}(X \times Y') \to \mathrm{CH}^m(X \times Y) \to \mathrm{CH}^m(X_{F(Y)})$$

is exact and ρ is a projector.

Now we consider the following commutative diagram

$$\begin{array}{ccc} \big(\rho^* \operatorname{CH}(X)\big) \otimes_\Lambda \operatorname{CH}(Y) & \longrightarrow & \operatorname{CH}(X \times Y) \circ \rho \\ \downarrow & & \downarrow \\ \operatorname{CH}(Y) & \longrightarrow & \operatorname{CH}(Y_{F(X)}) \end{array}$$

where the left homomorphism is induced by the augmentation map $\operatorname{CH}(X) \to \Lambda$. The right homomorphism is surjective by Corollary 2.2. As we checked right above, the top homomorphism is surjective in codimension m. Therefore the bottom homomorphism is also surjective in codimension m. □

The following statement is a particular case of [22, Theorem 2.11 (3 ⇒ 1)]:

LEMMA 2.9. – *Assume that X is A-trivial and $1 \in \deg \operatorname{CH}_0(X)$. Then for any F-variety Y, the change of field homomorphism $\operatorname{CH}(Y) \to \operatorname{CH}(Y_{F(X)})$ is an isomorphism.*

Proof. – To prove surjectivity, we note that any $y \in \operatorname{CH}(Y_{F(X)})$ is the image of some $\alpha \in \operatorname{CH}(X \times Y)$. If $x \in \operatorname{CH}_0(X)$ is an element of degree 1, then the correspondence $[X] \times x \in \operatorname{CH}^d(X \times X)$ satisfies by Lemma 2.6 the conditions of Lemma 2.1. Therefore $\alpha \circ ([X] \times x) \in \operatorname{CH}(X \times Y)$ is also mapped to $y \in \operatorname{CH}(Y_{F(X)})$. On the other hand, $\alpha \circ ([X] \times x) = [X] \times \alpha_*(x)$ is mapped to $\alpha_*(x)_{F(X)}$ and it follows that $\alpha_*(x)$ is an element of $\operatorname{CH}(Y)$ mapped to y.

Injectivity follows by specialization (see [10, §20.3] or [28]). □

COROLLARY 2.10. – *Assume that X is A-trivial. Then for any $l \in \deg \operatorname{CH}_0(X) \subset \Lambda$ and any F-variety Y, the image of $\operatorname{CH}(Y) \to \operatorname{CH}(Y_{F(X)})$ contains $l \operatorname{CH}(Y_{F(X)})$.*

Proof. – It suffices to consider the case $l = \deg x$ for a closed point $x \in X$. Let L be the residue field of x. The change of field homomorphism $\operatorname{CH}(Y_L) \to \operatorname{CH}(Y_{L(X)})$ is surjective by Lemma 2.9, and the transfer argument does the job. □

3. Abstract Rost motives

In this section, the coefficient ring Λ is $\mathbb{Z}_{(p)}$ (the ring of integers localized in a fixed prime p) or $\mathbb{F}_p$ (the finite field of p elements).

For any integer $n \geq 1$, an *abstract Rost motive of degree* $n+1$ with coefficients in Λ is a Chow motive $\mathcal{R}$ with coefficients in Λ living on a smooth complete geometrically irreducible variety X such that for any field extension L/F with $1 \in \deg \operatorname{CH}_0(X_L)$ one has $\mathcal{R}_L \simeq \Lambda \oplus \Lambda(b) \oplus \cdots \oplus \Lambda((p-1)b)$, where $b := (p^n - 1)/(p-1)$.

In particular, $\dim X \geq p^n - 1 = (p-1)b$. Pulling back the projector of $\mathcal{R}$ with respect to the diagonal of X, produces a 0-cycle class of degree p (cf. [17, Lemma 2.21]) showing that $\deg \operatorname{CH}_0(X) \supset p\Lambda$. It follows that the ideal $\deg \operatorname{CH}_0(X) \subset \Lambda$ of the coefficient ring Λ is equal either to $p\Lambda$ or to Λ.

The condition $1 \in \deg \operatorname{CH}_0(X_L)$ appearing in the definition means the same for $\Lambda = \mathbb{Z}_{(p)}$ as for $\Lambda = \mathbb{F}_p$. In Λ-free terms, it means that the variety X_L has a closed point of a prime to p degree.