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FINITENESS OF COMINUSCULE QUANTUM K-THEORY

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ABSTRACT. – The product of two Schubert classes in the quantum K-theory ring of a homogeneous space $X = G/P$ is a formal power series with coefficients in the Grothendieck ring of algebraic vector bundles on X. We show that if X is cominuscule, then this power series has only finitely many non-zero terms. The proof is based on a geometric study of boundary Gromov-Witten varieties in the Kontsevich moduli space, consisting of stable maps to X that take the marked points to general Schubert varieties and whose domains are reducible curves of genus zero. We show that all such varieties have rational singularities, and that boundary Gromov-Witten varieties defined by two Schubert varieties are either empty or unirational. We also prove a relative Kleiman-Bertini theorem for rational singularities, which is of independent interest. A key result is that when X is cominuscule, all boundary Gromov-Witten varieties defined by three single points in X are rationally connected.

RÉSUMÉ. – Le produit de deux classes de Schubert dans l'anneau de K-théorie quantique d'un espace homogène $X = G/P$ est une série formelle à coefficients dans l'anneau de Grothendieck des fibrés vectoriels algébriques au-dessus de X. Nous montrons que pour X cominuscule, cette série formelle n'a qu'un nombre fini de termes non nuls. La preuve repose sur une étude géométrique de certaines variétés de Gromov-Witten contenues dans le bord de l'espace de modules de Kontsevitch. Ces variétés paramètrent des applications stables à valeurs dans X, dont la courbe source est une union réductible de courbes rationnelles, et qui envoient les points marqués dans des sous-variétés de Schubert générales. Nous montrons que ces variétés de Gromov-Witten sont à singularités rationnelles et que celles définies par seulement deux sous-variétés de Schubert sont soit vides soit unirationnelles. Nous présentons également un énoncé relatif, de type Kleiman-Bertini pour les singularités rationnelles, d'intérêt indépendant. Un résultat-clé pour notre preuve est le fait que toutes les variétés de Gromov-Witten du bord de l'espace de modules de Kontsevitch, définies par trois variétés de Schubert ponctuelles dans X, sont rationnellement connexes.

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1. Introduction

The goal of this paper is to prove that any product of Schubert classes in the quantum K-theory ring of a cominuscule homogeneous space contains only finitely many non-zero terms.

Let $X = G/P$ be a homogeneous space defined by a semisimple complex Lie group G and a parabolic subgroup P, and let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of n-pointed stable maps to X of degree d, with total evaluation map $\mathrm{ev} : \overline{\mathcal{M}}_{0,n}(X,d) \to X^n$. Given Schubert varieties $\Omega_1, \dots, \Omega_n \subset X$ in general position, there is a *Gromov-Witten variety* $\mathrm{ev}^{-1}(\Omega_1 \times \cdots \times \Omega_n) \subset \overline{\mathcal{M}}_{0,n}(X,d)$, consisting of all stable maps that send the i-th marked point into Ω_i for each i. The Kontsevich space and its Gromov-Witten varieties are the foundation of the quantum cohomology ring of X, whose structure constants are the (cohomological) Gromov-Witten invariants, defined as the number of points in finite Gromov-Witten varieties. More generally, the *K-theoretic Gromov-Witten invariant* $I_d(\mathcal{O}_{\Omega_1}, \dots, \mathcal{O}_{\Omega_n})$ is defined as the sheaf Euler characteristic of $\mathrm{ev}^{-1}(\Omega_1 \times \cdots \times \Omega_n)$, which makes sense when this variety has positive dimension. The K-theoretic invariants are more challenging to compute, both because they are not enumerative, and also because they do not vanish for large degrees.

Assume for simplicity that P is a maximal parabolic subgroup of G, so that $H_2(X;\mathbb{Z}) = \mathbb{Z}$. The (small) *quantum K-theory ring* $\mathrm{QK}(X)$ is a formal deformation of the Grothendieck ring $K(X)$ of algebraic vector bundles on X, which as a group is defined by $\mathrm{QK}(X) = K(X) \otimes_{\mathbb{Z}} \mathbb{Z}[[q]]$. The product $\mathcal{O}_u \star \mathcal{O}_v$ of two Schubert structure sheaves is defined in terms of structure constants $N_{u,v}^{w,d} \in \mathbb{Z}$ such that

$$\mathcal{O}_u \star \mathcal{O}_v = \sum_{w,\, d \geq 0} N_{u,v}^{w,d}\, q^d\, \mathcal{O}_w\,.$$

In contrast to the quantum cohomology ring $\mathrm{QH}(X)$, the constants $N_{u,v}^{w,d}$ are not single Gromov-Witten invariants, but are defined as polynomial expressions of the K-theoretic Gromov-Witten invariants. A result of Givental asserts that $\mathrm{QK}(X)$ is an associative ring [13]. Since the K-theoretic Gromov-Witten invariants do not vanish for large degrees, the same might be true for the structure constants $N_{u,v}^{w,d}$, in which case the product $\mathcal{O}_u \star \mathcal{O}_v$ would be a power series in q with infinitely many non-zero terms. When X is a Grassmannian of type A, a combinatorial argument in [3] shows that this does not happen; all products in $\mathrm{QK}(X)$ are finite. In this paper we give a different geometric proof that shows more generally that all products in $\mathrm{QK}(X)$ are finite whenever X is a cominuscule homogeneous space. As a consequence, the quantum K-theory ring $\mathrm{QK}(X)$ provides a honest deformation of $K(X)$. The class of cominuscule varieties consists of Grassmannians of type A, Lagrangian Grassmannians, maximal orthogonal Grassmannians, and quadric hypersurfaces. In addition there are two exceptional varieties of type E called the Cayley plane and the Freudenthal variety.

Let $d_X(n)$ be the minimal degree of a rational curve passing through n general points of X. The numbers $d_X(n)$ for $n \leq 3$ have been computed explicitly in [7, 9], see the table in §4 below. Our main result is the following.

THEOREM 1. – *Let X be a cominuscule variety. Then $N_{u,v}^{w,d} = 0$ for $d > d_X(2)$.*

Theorem 1 holds also for the structure constants of the equivariant quantum K-theory ring $\mathrm{QK}_T(X)$, see Remark 5.3. The bound on d is sharp in the sense that $q^{d_X(2)}$ occurs in the square of a point in $\mathrm{QK}(X)$. In addition, this bound is also the best possible for the quantum cohomology ring $\mathrm{QH}(X)$ that does not depend on u, v, and w (cf. [12]).

Our proof uses that the structure constants $N_{u,v}^{w,d}$ can be rephrased as alternating sums of certain boundary Gromov-Witten invariants. Given a sequence $\mathbf{d} = (d_0, d_1, \dots, d_r)$ of effective degrees $d_i \in H_2(X;\mathbb{Z})$ such that $d_i > 0$ for $i > 0$ and $\sum d_i = d$, let $M_{\mathbf{d}} \subset \overline{\mathcal{M}}_{0,3}(X,d)$ be the closure of the locus of stable maps for which the domain is a chain of $r+1$ projective lines that map to X in the degrees given by $\mathbf{d}$, the first and second marked points belong to the first projective line, and the third marked point is on the last projective line. Then any constant $N_{u,v}^{w,d}$ can be expressed as an alternating sum of sheaf Euler characteristics of varieties of the form $\mathrm{ev}^{-1}(\Omega_1 \times \Omega_2 \times \Omega_3) \cap M_{\mathbf{d}}$. We use geometric arguments to show that the terms of this sum cancel pairwise whenever X is cominuscule and $d > d_X(2)$.

Set $Z_{\mathbf{d}} = \mathrm{ev}(M_{\mathbf{d}}) \subset X^3$. A key technical fact in our proof is that the general fibers of the map $\mathrm{ev} : M_{\mathbf{d}} \to Z_{\mathbf{d}}$ are rationally connected. Notice that these fibers are boundary Gromov-Witten varieties $\mathrm{ev}^{-1}(x \times y \times z) \cap M_{\mathbf{d}}$ defined by three single points in X, and the result generalizes the well known fact that there is a unique rational curve of degree d through three general points in the Grassmannian $\mathrm{Gr}(d, 2d)$ [5]. In the special case when $\mathbf{d} = (d)$ and $M_{\mathbf{d}} = \overline{\mathcal{M}}_{0,3}(X,d)$, it was shown in [3, 9] that the general fibers of ev are rational; our proof uses this case as well as Graber, Harris, and Starr's criterion for rational connectivity [14].

We also need to know that $M_{\mathbf{d}}$ has rational singularities. For this we prove a relative version of the Kleiman-Bertini theorem [18] for rational singularities. This theorem implies that any boundary Gromov-Witten variety in $\overline{\mathcal{M}}_{0,n}(X,d)$ has rational singularities, for any homogeneous space X. The Kleiman-Bertini theorem generalizes a result of Brion asserting that rational singularities are preserved when a subvariety of a homogeneous space is intersected with a general Schubert variety [1].

Finally, if Ω_1 and Ω_2 are Schubert varieties in general position in a homogeneous space X, we prove that $\mathrm{ev}_1^{-1}(\Omega_1) \cap M_{\mathbf{d}}$ is unirational and $\mathrm{ev}_1^{-1}(\Omega_1) \cap \mathrm{ev}_2^{-1}(\Omega_2) \cap M_{\mathbf{d}}$ is either empty or unirational. In particular, we have $I_d(\mathcal{O}_{\Omega_1}) = 1$ and $I_d(\mathcal{O}_{\Omega_1}, \mathcal{O}_{\Omega_2}) \in \{0,1\}$. This is done by showing that any Borel-equivariant map to a Schubert variety is locally trivial over the open cell. In particular, any single evaluation map $\mathrm{ev}_i : M_{\mathbf{d}} \to X$ is locally trivial.

Our paper is organized as follows. In Section 2 we prove the Kleiman-Bertini theorem for rational singularities and give a simple criterion for an equivariant map to be locally trivial. These results are applied to (boundary) Gromov-Witten varieties of general homogeneous spaces in Section 3. Section 4 proves some useful facts about images of Gromov-Witten varieties of cominuscule spaces, among them that the general fibers of $\mathrm{ev} : M_{\mathbf{d}} \to Z_{\mathbf{d}}$ are rationally connected. Finally, Section 5 applies these results to show that K-theoretic quantum products on cominuscule varieties are finite.

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2. A Kleiman-Bertini theorem for rational singularities

DEFINITION 2.1. – Let G be a connected algebraic group and X a G-variety. A *splitting* of the action of G on X is a morphism $s : U \to G$ defined on a dense open subset $U \subset X$, together with a point $x_0 \in U$, such that $s(x).x_0 = x$ for all $x \in U$. If a splitting exists, then we say that the action is *split* and that X is *G-split*.

Notice that any G-split variety contains a dense open orbit. Recall that if $X = G/P$ is a homogeneous space defined by a semisimple complex Lie group G and a parabolic subgroup P, then a *Schubert variety* in X is an orbit closure for the action of a Borel subgroup of G. Schubert varieties are our main examples of varieties with a split action.

PROPOSITION 2.2. – *Let G be a semisimple complex Lie group, $P \subset G$ a parabolic subgroup, and $X = G/P$ the corresponding homogeneous space with its natural G-action. Then X is G-split. Furthermore, if $B \subset G$ is a Borel subgroup and $\Omega \subset X$ is a B-stable Schubert variety, then Ω is B-split.*

Proof. – Let $\Omega \subset X$ be a B-stable Schubert variety, $\Omega^\circ \subset \Omega$ the B-stable open cell, and $x_0 \in \Omega^\circ$ any point. According to e.g., [20, Lemma 8.3.6] we can choose a unipotent subgroup $U \subset B$ such that the map $U \to \Omega^\circ$ defined by $g \mapsto g.x_0$ is an isomorphism. The inverse of this map is a splitting of the B-action on Ω. Since X is a Schubert variety, it follows that X is B-split and consequently G-split. □

Recall that a morphism $f : M \to X$ is a *locally trivial fibration* if each point $x \in X$ has an open neighborhood $U \subset X$ such that $f^{-1}(U) \cong U \times f^{-1}(x)$ and f is the projection to the first factor.

PROPOSITION 2.3. – *Let $f : M \to X$ be an equivariant map of irreducible G-varieties. Assume that X is G-split. Then f is a locally trivial fibration over the dense open G-orbit in X, and the fibers over this orbit are irreducible.*

Proof. – Let $x_0 \in U \subset X$ and $s : U \to G$ be a splitting of the G-action on X. Then the map $\varphi : U \times f^{-1}(x_0) \to f^{-1}(U)$ defined by $\varphi(x, y) = s(x).y$ is an isomorphism, with inverse given by $\varphi^{-1}(m) = (f(m), s(f(m))^{-1}.m)$. Since $f^{-1}(U) \cong U \times f^{-1}(x_0)$ is irreducible, so is $f^{-1}(x_0)$. □

In the rest of this section, a variety means a reduced scheme of finite type over an algebraically closed field of characteristic zero. An irreducible variety X has *rational singularities* if there exists a desingularization $\pi : \widetilde{X} \to X$ such that $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_{\widetilde{X}} = 0$ for all $i > 0$. An arbitrary variety has rational singularities if its irreducible components have rational singularities, are disjoint, and have the same dimension. Zariski's main theorem implies that any variety with rational singularities is normal. Notice also that if X and Y have rational singularities, then so does $X \times Y$. The converse is a special case of the following lemma of Brion [1, Lemma 3].

LEMMA 2.4 (Brion). – *Let Z and S be varieties and let $\pi : Z \to S$ be a morphism. If Z has rational singularities, then the same holds for the general fibers of π.*