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# ISOSPECTRALITY FOR QUANTUM TORIC INTEGRABLE SYSTEMS

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*To Peter Sarnak on his sixtieth birthday, with admiration*

ABSTRACT. – We give a full description of the semiclassical spectral theory of quantum toric integrable systems using microlocal analysis for Toeplitz operators. This allows us to settle affirmatively the isospectral problem for quantum toric integrable systems: the semiclassical joint spectrum of the system, given by a sequence of commuting Toeplitz operators on a sequence of Hilbert spaces, determines the classical integrable system given by the symplectic manifold and commuting Hamiltonians. This type of problem belongs to the realm of classical questions in spectral theory going back to pioneer works of Colin de Verdière, Guillemin, Sternberg and others in the 1970s and 1980s.

RÉSUMÉ. – Nous donnons une description complète du spectre de tout système intégrable torique quantique, au moyen de l'analyse microlocale des opérateurs de Toeplitz. Ceci résout la question de l'isospectralité pour cette classe de systèmes intégrables : le spectre semi-classique d'un système intégrable quantique torique détermine le système intégrable classique sous-jacent à symplectomorphisme près. Nous donnons aussi une description complète de la théorie spectrale semi-classique des systèmes intégrables toriques quantiques. Ces questions sont classiques en théorie spectrale et remontent aux travaux fondateurs de Colin de Verdière, Guillemin et Sternberg parmi d'autres dans les années 70 et 80.

## 1. Introduction

This paper gives a full description of the semiclassical spectral theory of *quantum toric integrable systems* in any finite dimension. The classical limits corresponding to quantum toric integrable systems are the so called symplectic toric manifolds or *toric systems*. Such a system consists of a compact symplectic $2n$-manifold equipped with $n$ commuting Hamiltonians $f_1, \dots, f_n$ with periodic flows. The paper combines geometric techniques from the theory of toric manifolds, in the complex-algebraic and symplectic settings, with recently developed microlocal analytic methods for Toeplitz operators.

As a consequence of the spectral theory we develop, we answer the *isospectrality* question for quantum toric integrable systems, in any finite dimension: the semiclassical joint spectrum of a quantum toric integrable system, given by a sequence of commuting Toeplitz operators acting on quantum Hilbert spaces, determines the classical system given by the symplectic manifold and Poisson commuting functions, up to symplectic isomorphisms. This type of symplectic isospectral problem belongs to the realm of classical questions in inverse spectral theory and microlocal analysis, going back to pioneer works of Colin de Verdière [14, 13] and Guillemin-Sternberg [37] in the 1970s and 1980s. Colin de Verdière's works are an important inspiration for the present paper.

The question of isospectrality in Riemannian geometry may be traced back to Weyl [72, 73] and is most well known thanks to Kac's article [41], who himself attributes the question to Bochner. Kac popularized the sentence: "*can one hear the shape of a drum?*," to refer to this type of isospectral problem. The spectral theory developed in this paper exemplifies a striking difference with Riemannian geometry, where this type of isospectrality rarely holds true, and suggests that *symplectic* invariants are much better encoded in spectral theory than *Riemannian* invariants. An approach to this problem for general integrable systems is suggested in the last two authors' article [60]. We refer to Section 8 for further remarks, and references, in these directions.

**Joint spectrum**

In order to state our results, let us introduce the required terminology. If $(M, \omega)$ is a symplectic manifold, a smooth map $\mu = (\mu_1, \ldots, \mu_n) : M \to \mathbb{R}^n$ is called a *momentum map* for a Hamiltonian $n$-torus action on $M$ if the Hamiltonian flows $t_j \mapsto \varphi_{\mu_j}^{t_j}$ are periodic of period 1, and pairwise commute :

$$\varphi_{\mu_j}^{t_j} \circ \varphi_{\mu_i}^{t_i} = \varphi_{\mu_i}^{t_i} \circ \varphi_{\mu_j}^{t_j},$$

so that they define an action of $\mathbb{R}^n/\mathbb{Z}^n$. If this action is effective and $M$ is compact, $2n$-dimensional and connected, we call $(M, \omega, \mu)$ a *symplectic toric manifold*.

By the Atiyah and Guillemin-Sternberg theorem, for any torus Hamiltonian action on a connected compact manifold, the image of the momentum map is a rational convex polytope [1, 36]. For a symplectic toric manifold, the momentum polytope $\Delta \subset \mathbb{R}^n$ has the additional property that for each vertex $v$ of $\Delta$, the primitive normal vectors to the facets meeting at $v$ form a basis of the integral lattice $\mathbb{Z}^n$. We call such a polytope a *Delzant polytope*.

A now standard procedure introduced by B. Kostant [43, 44, 45, 46] and J.-M. Souriau [62, 63] to quantize a symplectic compact manifold $(M, \omega)$ is to introduce a prequantum bundle $\mathscr{L} \to M$, that is a Hermitian line bundle with curvature $\frac{1}{i}\omega$ and a complex structure $j$ compatible with $\omega$. One then defines the quantum space as the space

$$\mathscr{H}_k := \mathrm{H}^0(M, \mathscr{L}^k)$$

of holomorphic sections of $\mathscr{L}^k$. The parameter $k$ is a positive integer, the semiclassical limit corresponds to the large $k$ limit. A description of this procedure, which is called *geometric quantization*, is given by Kostant and Pelayo in [47] from the angle of Lie theory and representation theory.

Not all symplectic manifolds have a complex structure or a prequantum bundle. However a symplectic toric manifold always admits a compatible complex structure, which is not unique. Furthermore a symplectic toric manifold $M$ with momentum map $\mu : M \to \mathbb{R}^n$ is prequantizable if and only if there exists $c \in \mathbb{R}^n$ such that the vertices of the polytope $\mu(M) + c$ belong to $2\pi\mathbb{Z}^n$ (see Section 3). If it is the case, the prequantum bundle is unique up to isomorphisms.

In many papers, a prequantum bundle is defined as a line bundle with curvature $\frac{1}{2\pi i}\omega$. With this normalization, the cohomology class of $\omega$ is integral and the prequantization condition for toric manifolds is that, up to translation, the momentum polytope has integral vertices. This normalization may look simpler than ours, which includes a $2\pi$-factor. Nevertheless, our choice is justified by the Weyl law. Indeed, with our normalization, the dimension of the quantum space $\mathcal{H}_k$ is

$$\left(\frac{k}{2\pi}\right)^n \operatorname{vol}(M, \omega) + \mathcal{O}(k^{n-1}).$$

Associated to such a quantization there is an algebra $\mathcal{T}(M, \mathcal{L}, j)$ of operators

$$T = (T_k : \mathcal{H}_k \to \mathcal{H}_k)_{k \in \mathbb{N}^*}$$

called *Toeplitz operators*. This algebra plays the same role as the algebra of semiclassical pseudodifferential operators for a cotangent phase space. Here the semiclassical parameter is $\hbar = 1/k$. A Toeplitz operator has a *principal symbol*, which is a smooth function on the phase space $M$. If $T$ and $S$ are Toeplitz operators, then $(T_k + k^{-1}S_k)_{k\in\mathbb{N}^*}$ is a Toeplitz operator with the same principal symbol as $T$. If $T_k$ is Hermitian (i.e., self-adjoint) for $k$ sufficiently large, then the principal symbol of $T$ is real-valued. Two Toeplitz operators $(T_k)_{k\in\mathbb{N}^*}$ and $(S_k)_{k\in\mathbb{N}^*}$ *commute* if $T_k$ and $S_k$ commute for every $k$.

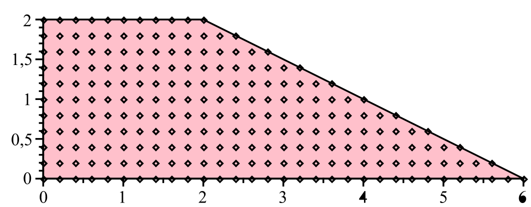


FIGURE 1. "Model image" of the spectrum of a normalized quantum toric integrable system.

We shall also need the following definitions. If $P_1, \dots, P_n$ are mutually commuting endomorphisms of a finite dimensional vector space, then the *joint spectrum of* $P_1, \dots, P_n$ is the set of $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that there exists a non-zero vector $v$ for which

$$P_j v = \lambda_j v,$$

for all $j = 1, \dots, n$. It is denoted by $\operatorname{JointSpec}(P_1, \dots, P_n)$. The *Hausdorff distance* between two subsets $A$ and $B$ of $\mathbb{R}^n$ is

$$d_H(A, B) := \inf\{\epsilon > 0 \mid A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\},$$

where for any subset $X$ of $\mathbb{R}^n$, the set $X_\epsilon$ is $X_\epsilon := \bigcup_{x\in X}\{m \in \mathbb{R}^n \mid \|x-m\| \leqslant \epsilon\}$. If $(A_k)_{k\in\mathbb{N}^*}$ and $(B_k)_{k\in\mathbb{N}^*}$ are sequences of subsets of $\mathbb{R}^n$, we say that

$$A_k = B_k + \mathcal{O}(k^{-\infty})$$

if $\mathrm{d}_H(A_k, B_k) = \mathcal{O}(k^{-N})$ for all $N \in \mathbb{N}^*$.

Our main result describes in full the joint spectrum of a quantum toric integrable system.

Theorem 1.1 (Joint Spectral Theorem). – *Let $(M, \omega, \mu : M \to \mathbb{R}^n)$ be a symplectic toric manifold equipped with a prequantum bundle $\mathcal{L}$ and a compatible complex structure $j$. Let $T_1, \dots, T_n$ be commuting Toeplitz operators of $\mathcal{T}(M, \mathcal{L}, j)$ whose principal symbols are the components of $\mu$. Then the joint spectrum of $T_1, \dots, T_n$ satisfies*

$$\mathrm{JointSpec}(T_1, \dots, T_n) = g\Big(\Delta \cap \Big(v + \frac{2\pi}{k}\mathbb{Z}^n\Big);\, k\Big) + \mathcal{O}(k^{-\infty})$$

*where $\Delta = \mu(M)$, $v$ is any vertex of $\Delta$ and $g(\cdot; k) : \mathbb{R}^n \to \mathbb{R}^n$ admits a $\mathrm{C}^\infty$-asymptotic expansion of the form*

$$g(\cdot; k) = \mathrm{Id} + k^{-1}g_1 + k^{-2}g_2 + \cdots$$

*where each $g_j : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Moreover, for all sufficiently large $k$, the multiplicity of the eigenvalues of* $\mathrm{JointSpec}(T_1, \dots, T_n)$ *is* 1*, and there exists a small constant $\delta > 0$ such that each ball of radius $\frac{\delta}{k}$ centered at an eigenvalue contains precisely only that eigenvalue.*

Thus the joint spectrum of a quantum toric integrable system can be obtained by taking the $k^{-1}\mathbb{Z}^n$ lattice points in a polytope $\Delta$ (as in Figure 1), and applying a small smooth deformation $g$ (as in Figure 2).

### Isospectrality

We present next the isospectral theorem for toric systems. An easy consequence of the previous theorem is that the momentum polytope $\Delta$ is the Hausdorff limit of the joint spectrum of the quantum system, that is $\Delta$, consists of the $\lambda \in \mathbb{R}^n$ such that for any neighborhood $U$ of $\lambda$, $U \cap \mathrm{JointSpec}(T_{1,k}, \dots, T_{n,k}) \neq \varnothing$ when $k$ is sufficiently large.

Recall that two symplectic toric manifolds $(M, \omega, \mu)$ and $(M', \omega', \mu')$ are *isomorphic* if there exists a symplectomorphism $\varphi : M \to M'$ such that

$$\varphi^*\mu' = \mu.$$

By the Delzant classification theorem [21], a symplectic toric manifold is determined up to isomorphism by its momentum polytope. Furthermore, for any Delzant polytope $\Delta$, Delzant constructed in [21] a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mu_\Delta)$ with momentum polytope $\Delta$. Now we are ready to state our isospectral theorem (see Figure 2 for an illustration of the semiclassical joint spectrum).

Corollary 1.2 (Isospectral Theorem). – *Let $(M, \omega, \mu : M \to \mathbb{R}^n)$ be a symplectic toric manifold equipped with a prequantum bundle $\mathcal{L}$ and a compatible complex structure $j$. Let $T_1, \dots, T_n$ be commuting Toeplitz operators of $\mathcal{T}(M, \mathcal{L}, j)$ whose principal symbols are the components of $\mu$. Then*

$$\Delta := \lim_{k\to\infty} \mathrm{JointSpec}(T_1, \dots, T_n)$$