ISSN 0012-9593

quatrième série - tome 46 *fascicule 6* *novembre-décembre 2013*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Carlos MATHEUS & Carlos G. MOREIRA & Enrique R. PUJALS

Axiom A versus Newhouse phenomena for Benedicks-Carleson toy models

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 46, 2013, p. 857 à 878

AXIOM A VERSUS NEWHOUSE PHENOMENA FOR BENEDICKS-CARLESON TOY MODELS

BY CARLOS MATHEUS, CARLOS G. MOREIRA AND ENRIQUE R. PUJALS

ABSTRACT. – We consider a family of planar systems introduced in 1991 by Benedicks and Carleson as a toy model for the dynamics of the so-called Hénon maps. We show that Smale's Axiom A property is C^1-dense among the systems in this family, despite the existence of C^2-open subsets (closely related to the so-called Newhouse phenomena) where Smale's Axiom A is violated. In particular, this provides some evidence towards Smale's conjecture that Axiom A is a C^1-dense property among surface diffeomorphisms.

The basic tools in the proof of this result are: (1) a recent theorem of Moreira saying that stable intersections of dynamical Cantor sets (one of the main obstructions to Axiom A property for surface diffeomorphisms) can be destroyed by C^1-perturbations; (2) the good geometry of the dynamical critical set (in the sense of Rodriguez-Hertz and Pujals) thanks to the particular form of Benedicks-Carleson toy models.

RÉSUMÉ. – Nous considérons une famille de systèmes introduite en 1991 par Benedicks et Carleson comme un modèle jouet pour la dynamique des applications d'Hénon. Nous montrons que l'axiome A de Smale est une propriété C^1-dense parmi les systèmes dans cette famille, même si nous trouvons aussi des ensembles C^2-ouverts (liés au phénomène de Newhouse) où l'axiome A de Smale n'est pas satisfait. En particulier, notre résultat soutient la conjecture de Smale selon laquelle l'axiome A est une propriété C^1-dense parmi les difféomorphismes de surfaces.

Les outils utilisés dans la preuve de notre résultat sont : (1) un théorème récent de Moreira qui dit que les intersections stables des ensembles de Cantor dynamiques (une des obstructions majeures à l'axiome A pour les difféomorphismes de surfaces) peuvent être enlevées par des perturbations C^1-petites ; (2) la bonne géométrie de l'ensemble de points critiques dynamiques (au sens de Rodriguez-Hertz et Pujals) due à la forme particulière des modèles jouets de Benedicks-Carleson.

1. Introduction

Uniform hyperbolicity (Smale's Axiom A property) has been a long standing paradigm of complete dynamical description: any dynamical system such that the tangent bundle over its limit set (i.e., the set of accumulation points of all orbits) splits into two complementary subbundles which are uniformly forward (respectively backward) contracted by the tangent map can be completely described from a geometrical and topological point of view.

Nevertheless, uniform hyperbolicity is a property less universal than it was initially thought: *there are non-empty open sets in the space of dynamics containing only non-hyperbolic systems.* Actually, Newhouse showed that for smooth surface diffeomorphisms, the unfolding of a *homoclinic tangency* (a non transversal intersection of stable and unstable manifolds of a periodic point) generates non-empty open sets of diffeomorphisms whose limit sets are non-hyperbolic (see [9], [11], [12]).

It is important to say that a homoclinic tangency is (locally) easily destroyed by small perturbation of the invariant manifolds. To get open sets of diffeomorphisms with persistent homoclinic tangencies, Newhouse considers certain systems where the homoclinic tangency is associated to an invariant hyperbolic set with large fractal dimension. In particular, he studied the intersection of the local stable and unstable manifolds of a hyperbolic set (for instance, a classical horseshoe), which, roughly speaking, can be visualized as a product of two Cantor sets whose thicknesses are large. Newhouse's construction depends on how this fractal invariant varies with perturbations of the dynamics, and actually this is the main reason that his construction works in the C^2-topology. In fact, Newhouse argument is based on the continuous dependence of the thickness with respect to C^2-perturbations. A similar construction in the C^1-topology leading to same phenomena is unknown (indeed, some results in the *opposite* direction can be found in [18] and [8]). In this setting, denoting by $\mathrm{Diff}^r(M^n)$ the set of C^r-diffeomorphisms of a compact n-dimensional manifold M^n (without boundary), it was *implicitly* conjectured by Smale (cf. [17], Problems (6.10), item (a), at page 779) that

Axiom A surface diffeomorphisms are C^1 open and dense in $\mathrm{Diff}^1(M^2)$.

This question is explicitly called *Smale's conjecture* in [1].

In the present paper, we consider a special set of maps acting on a two dimensional rectangle, firstly introduced by Benedicks and Carleson as a *toy model* for the so-called *Hénon maps*. For this special type of systems, we show that, if one deals with C^2-topology, there are non-empty open sets of diffeomorphisms which are not hyperbolic, while in the C^1-topology, the Axiom A property is open and dense.

Before proceeding further, let us briefly recall some features of Hénon maps and Benedicks-Carleson toy models.

A typical family where the Newhouse's phenomena hold is the so called Hénon maps. In fact, it was proved in [19] that, for certain parameter of this family, the unfolding of a tangency leads to a non-empty open set of non-hyperbolic diffeomorphisms.

On the other hand, numerical simulations indicate that the attractor of the Hénon map (i.e., the closure of the unstable manifold of its fixed saddle point) has the structure of the product of a line segment and a Cantor set with small dimension (when a certain parameter b is close to zero). Although it is a great oversimplification (and many of the later difficulties on the analysis of Hénon attractors arise because of the roughness of such approximation), this idea gives a very good understanding of the geometry of the Hénon map. As a guide to what follows, it is worth to point out that Benedicks and Carleson [3, Section 3, p. 89] have constructed a model where the point moves on a pure product space $(-1,1)\times\mathcal{K}$ where $\mathcal{K}$ is the Cantor set obtained by repeated iteration of the division proportions $(b, 1-2b, b)$ (i.e., $\mathcal{K} = \bigcap_{n\geq 0} A^{-n}([0,b] \cup [1-b,1])$ where $A|_{[0,b]}(x) = x/b$ and $A|_{[1-b,1]}(x) = (1-x)/b$), and the dynamics on $(-1,1)$ is given by a family of quadratic maps: in fact, the dynamical system on $(-1,1)$ acts as a movement on a fan of lines, where each line has its own x-evolution, while it is contracted in the y-direction (see Figure 1).

More precisely, consider a one parameter C^r-family $\{f(x,y)\}_{y\in[0,1]}$ (here x is the variable and y is the parameter) such that, for each fixed parameter $y \in [0,1]$,

$$f(.,y) : [-1,1] \to [-1,1]$$

is a C^r-unimodal map (with respect to the variable x) verifying that 0 is a critical point and $f(0,y)$ is the maximum value of $f(.,y)$ for all $y \in [0,1]$. We denote by $\mathcal{U}^r$ the set of C^r-families of C^r-unimodal maps satisfying the conditions stated above.

Let $k : [0,a] \cup [b,1] \to [0,1]$ be a C^r function such that $k(0) = 0 = k(1)$, $k(a) = 1 = k(b)$ and $|k'| > \gamma > 1$. Put

$$K(x,y) = \begin{cases} K_+(y) \text{ if } x > 0, \\ K_-(y) \text{ if } x < 0, \end{cases}$$

where $K_+ = (k_{/[0,a]})^{-1}, K_- = (k_{/[b,1]})^{-1}$.

The bulk of this article is the study of the dynamics of Benedicks-Carleson toy models $F : ([-1,1] \setminus \{0\}) \times [0,1] \to [-1,1] \times [0,1]$ given by

(1) $$F(x,y) = (f(x,y), K(x,y)) = (f(x,y), K_{\mathrm{sgn}(x)}(y)).$$

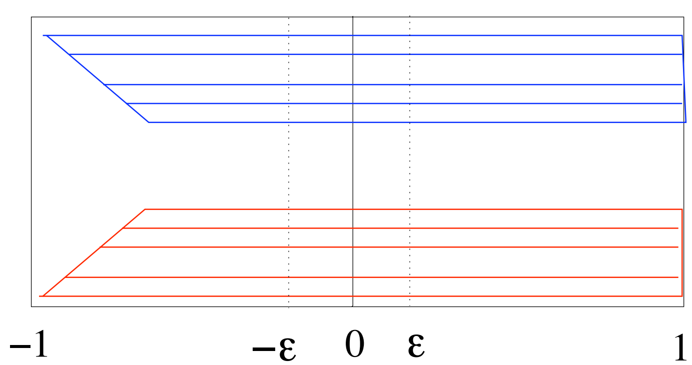

FIGURE 1. Dynamics of $F(x,y) = (1 - a(y)x^2, K_{\mathrm{sgn}(x)}(y))$ with $a : [0,1] \to (0,2]$.

We denote by $\mathcal{D}^r$ the set of such maps F (with $f(.,y) \in \mathcal{U}^r$ and $k \in C^r$) endowed with the C^r-topology. Since the line $\{x = 0\}$ is a discontinuity line of any $F \in \mathcal{D}^r$, the maps F are C^r-diffeomorphisms only on $([-1,1] - \{0\}) \times [0,1]$, and we will explain in Section 2 the exact definition of the C^r-topology. Although this is not specially hard to do, we prefer to postpone it (where we also revise the notion of hyperbolic sets of F) to avoid the appearance of unnecessary technicalities in this introductory section.

At this point, we are ready to state our main results:

THEOREM A. – *For $r \geq 2$, there exists a non-empty open set $\mathcal{N} \subset \mathcal{D}^r$ such that no $F \in \mathcal{N}$ is Axiom A. Moreover, there exists a residual set $\mathcal{R} \subset \mathcal{N}$ such that any $F \in \mathcal{R}$ has infinitely many periodic sinks.*

On the other hand, in the C^1-topology, the opposite statement holds:

THEOREM B. – *There exists an open and dense set $\mathcal{V} \subset \mathcal{D}^1$ such that every $F \in \mathcal{V}$ is Axiom A.*

Concerning the proof of these results, a fundamental role will be played by certain points in the line $\{x = 0\}$:

DEFINITION 1. – *Given $F \in \mathcal{D}^r$, consider $k : [0, a] \cup [b, 1] \to [0, 1]$ the Cantor map related to F and denote by $\mathcal{K}_0$ the Cantor set induced by k. For any $y \in \mathcal{K}_0$, we call*

$$c_y^{\pm} = (0^{\pm}, y)$$

a critical point *of F.*

From the technical point of view, it is important to introduce the points $c_y^{\pm} = (0^{\pm}, y)$ because we can extend F to them (via the formula $F(0^{\pm}, y) = (f(0, y), K_{\pm}(y))$), so that F becomes defined on a compact set. Of course, we have to pay the price that this extension of F is no longer continuous. In particular, let us make a few comments about the orbits and the meaning of the non-wandering set $\Omega(F)$ of this extension of F. While the orbits do not intersect the line $\{x = 0\}$, we have nothing to say. On the other hand, when the iterate $(0, y) = F(z, w)$ of a point $(z, w) \in ([-1, 1] - \{0\}) \times [0, 1]$ hits the line $\{x = 0\}$, we will consider that *both* points $(0^{\pm}, y)$ make part of the orbit of (z, w). Finally, we say that a point (z, w) is non-wandering when any neighborhood U of (z, w) has some iterate $F^n(U)$ such that $F^n(U) \cap U \neq \varnothing$. Here, a small neighborhood of a point $(z, w) \in ([-1, 1] - \{0\}) \times [0, 1]$ is a small standard (Euclidean) neighborhood, while a small neighborhood of the point $(0^+, y)$, resp. $(0^-, y)$, is a "half-neighborhood" obtained from the intersection of $[0, 1] \times [0, 1]$, resp. $[-1, 0] \times [0, 1]$, with a small standard (Euclidean) neighborhood of $(0, y)$.

The relevance of the concept of critical point becomes clear from the following simple (but conceptually important) remark:

REMARK 1.1. – *It follows from the definition that, if $c_y^{\pm} \in \Omega(F)$ and $c_y^{\pm}$ is not a periodic sink, then $\Omega(F)$ is not hyperbolic in the sense of Definition 3 below. This fact should be compared to the notion of* dynamical critical points *of* [15] *and its role as the obstruction to the presence of hyperbolicity/domination in dissipative compact invariant sets of surface diffeomorphisms.*

Closing this introduction, we give the organization of the paper:

- In Section 3, we follow the same ideas of Newhouse to construct a C^2-open set $\mathcal{N}$ where the critical points cannot be removed from the limit set, so that the proof of Theorem A can be derived from the combination of this fact and Remark 1.1.