ISSN 0012-9593

quatrième série - tome 46 fascicule 6 novembre-décembre 2013

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Boris ADAMCZEWSKI & Jason P. BELL

Diagonalization and Rationalization of algebraic Laurent series

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 46, 2013, p. 963 à 1004

DIAGONALIZATION AND RATIONALIZATION OF ALGEBRAIC LAURENT SERIES

BY Boris ADAMCZEWSKI AND Jason P. BELL

ABSTRACT. – We prove a quantitative version of a result of Furstenberg [20] and Deligne [14] stating that the diagonal of a multivariate algebraic power series with coefficients in a field of positive characteristic is algebraic. As a consequence, we obtain that for every prime p the reduction modulo p of the diagonal of a multivariate algebraic power series f with integer coefficients is an algebraic power series of degree at most p^A and height at most Ap^A, where A is an effective constant that only depends on the number of variables, the degree of f and the height of f. This answers a question raised by Deligne [14].

RÉSUMÉ. – Nous démontrons une version quantitative d'un résultat de Furstenberg [20] et Deligne [14] : la diagonale d'une série formelle algébrique de plusieurs variables à coefficients dans un corps de caractéristique non nulle est une série formelle algébrique d'une variable. Comme conséquence, nous obtenons que, pour tout nombre premier p, la réduction modulo p de la diagonale d'une série formelle algébrique de plusieurs variables f à coefficients entiers est une série formelle algébrique de degré au plus p^A et de hauteur au plus Ap^A, où A est une constante effective ne dépendant que du nombre de variables, du degré de f et de la hauteur de f. Cela répond à une question soulevée par Deligne [14].

1. Introduction

A very rich interplay between arithmetic, geometry, transcendence and combinatorics arises in the study of homogeneous linear differential equations and especially of those that "come from geometry" and the related study of Siegel G-functions (see for instance [4, 16, 22, 30, 31, 32] for discussions that emphasize these different aspects). As an illustration, let us recall a few of the many classical results attached to the differential equation

$$t(t-1)y''(t) + (2t-1)y'(t) + \frac{1}{4}y(t) = 0\,.$$

The first author was supported by the project Hamot, ANR 2010 BLAN-0115-01. The second author was supported by NSERC grant 31-611456.

- This differential equation has (up to multiplication by a scalar) a unique solution that is holomorphic at the origin. This solution is a classical hypergeometric function

$$f_1(t) := {}_2F_1(1/2, 1/2;\, 1;\, t) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n}^2 t^n \in \mathbb{Q}[[t]]\,.$$

- It has also the following integral form:

$$f_1(t) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - t\sin^2\theta}}\,.$$

  In particular,

$$\pi f_1(t) = \int_1^{+\infty} \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

  is an elliptic integral and a period in the sense of Kontsevich and Zagier [22].

- For nonzero algebraic numbers t in the open unit disc, $f_1(t)$ and $\pi f_1(t)$ are both known to be transcendental (see for instance the complete survey [33]). In particular, the function f_1 is a transcendental function over the field $\mathbb{Q}(t)$.

- This differential equation comes from geometry: it is the Picard–Fuchs equation of the Legendre family of elliptic curves $\mathcal{E}_t$ defined by the equation $y^2 = x(x-1)(x-t)$.

- The Taylor expansion of f_1 has almost integer coefficients. In particular,

$$f_1(16t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 t^n \in \mathbb{Z}[[t]]$$

  corresponds to a classical generating function in enumerative combinatorics (associated for instance with the square lattice walks that start and end at origin).

A remarkable result is that, by adding variables, we can see f_1 as arising in a natural way from a much more elementary function, namely a rational function. Indeed, let us consider the rational function

$$R(x_1, x_2, x_3, x_4) := \frac{2}{2 - x_1 - x_2} \cdot \frac{2}{2 - x_3 - x_4}\,.$$

Then R can be expanded as

$$R = \sum_{(i_1,i_2,i_3,i_4)\in\mathbb{N}^4} a(i_1, i_2, i_3, i_4)\, x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$$

$$= \sum_{(i_1,i_2,i_3,i_4)\in\mathbb{N}^4} 2^{-(i_1+i_2+i_3+i_4)} \binom{i_1+i_2}{i_1} \binom{i_3+i_4}{i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}\,.$$

Collecting all the diagonals terms, we easily get that

$$\Delta(R) := \sum_{n=0}^{\infty} a(n,n,n,n) t^n = f_1(t)\,.$$

More formally, given a field K and a multivariate power series

$$f(x_1,\dots,x_n) := \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} a(i_1,\dots,i_n)x_1^{i_1}\cdots x_n^{i_n}$$

with coefficients in K, we define the *diagonal* $\Delta(f)$ of f as the one variable power series

$$\Delta(f)(t) := \sum_{n=0}^{\infty} a(n,\dots,n)t^n \in K[[t]]\,.$$

Another classical example which emphasizes the richness of diagonals is the following. The power series

$$f_2(t) := \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n \in \mathbb{Z}[[t]]$$

is a well-known transcendental G-function that appears in Apéry's proof of the irrationality of $\zeta(3)$ (see [18]). It is also known to satisfy the Picard–Fuchs equation associated with a one-parameter family of K_3 surfaces [6]. Furthermore, a simple computation shows that f_2 is the diagonal of the five-variable rational function

$$\frac{1}{1-x_1}\cdot\frac{1}{(1-x_2)(1-x_3)(1-x_4)(1-x_5)-x_1x_2x_3} \in \mathbb{Z}[[x_1,\dots,x_5]]\,.$$

These two examples actually reflect a general phenomenon. In the case where $K=\mathbb{C}$, diagonalization may be nicely visualized thanks to Deligne's formula [14] via contour integration over a vanishing cycle. Formalizing this in terms of the Gauss–Manin connection and De Rham cohomology groups, and using a deep result of Grothendieck, one can prove that the diagonal of any algebraic power series with algebraic coefficients is a Siegel G-function that comes from geometry, that is, one which satisfies the Picard–Fuchs type equation associated with some one-parameter family of algebraic varieties [4, 10]. As claimed by the Bombieri–Dwork conjecture, this is a picture expected for all G-functions. Diagonals of algebraic power series with coefficients in $\overline{\mathbb{Q}}$ thus appear to be a distinguished class of G-functions. Originally introduced in the study of Hadamard products [7], diagonals have since been studied by many authors and for many different reasons [8, 9, 10, 14, 15, 20, 23, 24, 30, 31].

Remark 1.1. – The same power series may well arise as the diagonal of different rational functions, but it is expected that the underlying families of algebraic varieties should be connected in some way, such as *via* the existence of some isogenies (see the discussion in [10]). For instance, $f_1(t)$ is also the diagonal of the three-variable rational function

$$\frac{4}{4-(x_1+x_2)(1+x_3)}\,,$$

while $f_2(t)$ is also the diagonal of the six-variable rational function

$$\frac{1}{(1-x_1x_2)(1-x_3-x_4-x_1x_3x_4)(1-x_5-x_6-x_2x_5x_6)}\,.$$

When K is a field of positive characteristic, the situation is completely different as shown in the following nice result.

Definition 1.1. – A power series $f(x_1,\dots,x_n)\in K[[x_1,\dots,x_n]]$ is said to be *algebraic* if it is algebraic over the field of rational functions $K(x_1,\dots,x_n)$, that is, if there exist polynomials $A_0,\dots,A_m\in K[x_1,\dots,x_n]$, not all zero, such that $\sum_{i=0}^m A_i(x_1,\dots,x_n)f(x_1,\dots,x_n)^i=0$. The *degree* of f is the minimum of the positive integers m for which such a relation holds. The (naive) *height* of f is defined as the minimum of the heights of the nonzero polynomials $P(Y)\in K[x_1,\dots,x_n][Y]$ that vanish at f, or equivalently, as the height of the minimal polynomial of f. The height of a polynomial $P(Y)\in K[x_1,\dots,x_n][Y]$ is the maximum of the total degrees of its coefficients.

Theorem 1.1 (Furstenberg–Deligne). – *Let K be a field of positive characteristic. Then the diagonal of an algebraic power series in $K[[x_1,\dots,x_n]]$ is algebraic.*

Furstenberg [20] first proved the case where f is a rational power series and Deligne [14] extended this result to algebraic power series by using tools from arithmetic geometry. Some elementary proofs have then been worked out independently by Denef and Lipshitz [15], Harase [21], Sharif and Woodcock [28] (see also Salon [26]). The present work is mainly motivated by the following consequence of Theorem 1.1. Given a prime number p and a power series $f(x):=\sum_{n=0}^\infty a(n)x^n\in\mathbb{Z}[[x]]$, we denote by $f_{|p}$ the reduction of f modulo p, that is

$$f_{|p}(x):=\sum_{n=0}^{\infty}(a(n) \bmod p)x^n\in\mathbb{F}_p[[x]]\,.$$

Theorem 1.1 implies that if $f(x_1,\dots,x_n)\in\mathbb{Z}[[x_1,\dots,x_n]]$ is algebraic over $\mathbb{Q}(x_1,\dots,x_n)$, then $\Delta(f)_{|p}$ is algebraic over $\mathbb{F}_p(x)$ for every prime p. In particular, both the transcendental functions f_1 and f_2 previously mentioned have the remarkable property of having algebraic reductions modulo p for every prime p.

It now becomes very natural to ask how the complexity of the algebraic function $\Delta(f)_{|p}$ may increase when p runs along the primes. A common way to measure the complexity of an algebraic power series is to estimate its degree and its height. Deligne [14] obtained a first result in this direction by proving that if $f(x,y)\in\mathbb{Z}[[x,y]]$ is algebraic, then, for all but finitely many primes p, $\Delta(f)_{|p}$ is an algebraic power series of degree at most Ap^B, where A and B do not depend on p but only on geometric quantities associated with f. He also suggested that a similar bound should hold for the diagonal of algebraic power series in $\mathbb{Z}[[x_1,\dots,x_n]]$. Our main aim is to provide the following answer to the question raised by Deligne.

Theorem 1.2. – *Let $f(x_1,\dots,x_n)\in\mathbb{Z}[[x_1,\dots,x_n]]$ be an algebraic power series with degree at most d and height at most h. Then there exists an effective constant $A:=A(n,d,h)$ depending only on n, d and h, such that $\Delta(f)_{|p}$ has degree at most p^A and height at most Ap^A, for every prime number p.*

Theorem 1.2 is derived from the following quantitative version of the Furstenberg–Deligne theorem.