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MICROLOCAL LIMITS OF PLANE WAVES AND EISENSTEIN FUNCTIONS

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Abstract. – We study microlocal limits of plane waves on noncompact Riemannian manifolds (M, g) which are either Euclidean or asymptotically hyperbolic with curvature -1 near infinity. The plane waves $E(z, \xi)$ are functions on M parametrized by the square root of energy z and the direction of the wave, ξ, interpreted as a point at infinity. If the trapped set K for the geodesic flow has Liouville measure zero, we show that, as $z \to +\infty$, $E(z, \xi)$ microlocally converges to a measure μ_ξ, in average on energy intervals of fixed size, $[z, z+1]$, and in ξ. We express the rate of convergence to the limit in terms of the classical escape rate of the geodesic flow and its maximal expansion rate—when the flow is Axiom A on the trapped set, this yields a negative power of z. As an application, we obtain Weyl type asymptotic expansions for local traces of spectral projectors with a remainder controlled in terms of the classical escape rate.

Résumé. – Dans ce travail, nous étudions les mesures microlocales des fonctions de type ondes planes sur des variétés non compactes (M, g) qui, près de l'infini, sont euclidiennes ou asymptotiquement hyperboliques avec courbure -1. Les ondes planes $E(z, \xi)$ sont des fonctions sur M paramétrées par la racine carrée de l'énergie z et la direction ξ de l'onde, interprétée comme un point à l'infini. Si l'ensemble capté K pour le flot géodésique est de mesure de Liouville nulle, nous montrons que, quand $z \to +\infty$, $E(z, \xi)$ converge microlocalement vers une certaine mesure μ_ξ, en moyenne en ξ et en énergie z sur des intervalles de taille fixe. On exprime la vitesse de convergence vers la limite en fonction de la vitesse de fuite du flot géodésique et de son taux maximal d'expansion. Quand le flot est Axiom A sur K, la vitesse de convergence est une puissance négative de z. Enfin, en guise d'application, nous donnons des développements asymptotiques de type Weyl à plusieurs termes pour les traces locales de projecteurs spectraux, avec un reste dépendant de la vitesse de fuite du flot.

For a compact Riemannian manifold (M, g) of dimension d whose geodesic flow is ergodic with respect to the Liouville measure μ_L, *quantum ergodicity* (QE) of eigenfunctions [48, 58, 7] states that any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of eigenfunctions of the Laplacian with eigenvalues z_j^2, has a density one subsequence (e_{j_k}) that converges microlocally to μ_L in the following sense: for each symbol $a \in C^\infty(T^*M)$ of order zero,

$$\langle \mathrm{Op}_{h_{j_k}}(a) e_{j_k}, e_{j_k} \rangle_{L^2(M)} \to \frac{1}{\mu_L(S^*M)} \int_{S^*M} a \, d\mu_L. \tag{1.1}$$

Here S^*M stands for the unit cotangent bundle, $\mathrm{Op}_h(a)$ denotes the pseudodifferential operator obtained by quantizing a (see Section 3.1), and we put $h_j = z_j^{-1}$. The proof uses the following integrated form of quantum ergodicity [25]:

$$h^{d-1} \sum_{h^{-1} \leq z_j \leq h^{-1}+1} \left| \langle \mathrm{Op}_h(a) e_j, e_j \rangle_{L^2} - \frac{1}{\mu_L(S^*M)} \int_{S^*M} a \, d\mu_L \right| \to 0 \quad \text{as } h \to 0. \tag{1.2}$$

See Appendix D for a short self-contained proof of this result using the methods of this paper.

In the present paper, we consider a non-compact complete Riemannian manifold (M, g) and show that generalized eigenfunctions of the Laplacian on M known in scattering theory as *distorted plane waves* or *Eisenstein functions*, converge microlocally on average, similarly to (1.2), with the limiting measure μ_ξ depending on the direction of the plane wave ξ—see Theorem 1. We also give estimates on the rate of convergence in terms of classical quantities defined from the geodesic flow on M—see Theorem 2.

Our microlocal convergence of plane waves is similar in spirit to the QE Results (1.1) and (1.2). However, unlike the case of QE where *ergodicity* of the geodesic flow is essential, our result is based on a different phenomenon, roughly described as *dispersion* of plane waves. This difference manifests itself in the proofs as follows: instead of averaging an observable along the geodesic flow as in the standard proof of quantum ergodicity, we propagate it. See Section 2 for an outline of the proofs of Theorems 1 and 2.

Geometric assumptions near infinity. – The manifold M has dimension $d = n + 1$. For our results to hold, we need to make several assumptions on the geometry of (M, g) near infinity and on the spectral decomposition of its Laplacian Δ. They are listed in Section 4 and we check in Sections 6 and 7 that they are satisfied in each of the following two cases:

1. there exists a compact set $K_0 \subset M$ such that $(M \setminus K_0, g)$ is isometric to $\mathbb{R}^{n+1} \setminus B(0, R_0)$ with the Euclidean metric g_0 for some $R_0 > 0$; here $B(0, R_0)$ denotes the ball centered at 0 of radius R_0,
2. (M, g) is an *asymptotically hyperbolic* manifold in the sense that it admits a smooth compactification $\overline{M}$ and there exists a smooth boundary defining function x such that in a collar neighborhood of the boundary $\partial\overline{M}$, the metric has the form

$$g = \frac{dx^2 + h(x)}{x^2}, \tag{1.3}$$

   where $h(x)$ is a smooth 1-parameter family of metrics on $\partial\overline{M}$ for $x \in [0, \varepsilon)$. We further assume that g has sectional curvature -1 in a neighborhood of $\partial\overline{M}$.

In case (1), we call (M, g) *Euclidean near infinity*, while in case (2), we call it *hyperbolic near infinity*. Case (2) in particular includes convex co-compact hyperbolic quotients $\Gamma \setminus \mathbb{H}^{n+1}$—see Appendix A. Other possible geometries are discussed in Section 2.1.

Distorted plane waves/Eisenstein functions. – Let Δ be the (nonnegative) Laplace-Beltrami operator on M. In the study of the relation between classical dynamics and high energy behavior it is natural to use the semiclassically rescaled operator $h^2\Delta$, with $h > 0$ small parameter tending to zero.

The operator $h^2\Delta$ has continuous spectrum on a half-line $[c_0 h^2, \infty)$ (here c_0 is 0 for the Euclidean and $n^2/4$ for the hyperbolic case) and possibly a finite number of eigenvalues in $(0, c_0 h^2)$. The continuous spectrum is parametrized by *distorted plane waves* (or *Eisenstein functions* in the hyperbolic case) $E_h(\lambda,\xi) \in C^\infty(M)$, satisfying for $\lambda \in \mathbb{R}$,

$$(h^2(\Delta - c_0) - \lambda^2)E_h(\lambda,\xi) = 0. \tag{1.4}$$

Because of the h-rescaling, the effective spectral parameter is λ/h. Here ξ lies on the boundary $\partial\overline{M}$ of a compactification $\overline{M}$ of M. We can think of an element of $\partial\overline{M}$ as the direction of escape to infinity for a non-trapped geodesic; then ξ is the direction of the outgoing part of the plane wave $E_h(\lambda,\xi)$ at infinity.

For instance, in the case of manifolds Euclidean near infinity, $c_0 = 0$, $\partial\overline{M} = \mathbb{S}^n$ is the sphere, and for $m \in M \setminus K_0 \simeq \mathbb{R}^{n+1} \setminus B(0,R_0)$,

$$E_h(\lambda,\xi;m) = e^{\frac{i\lambda}{h}\xi\cdot m} + E_{\mathrm{inc}},$$

where E_{inc} is incoming in the sense that there exists $f \in C^\infty(S^n)$ such that $[E_{\mathrm{inc}}(\lambda,\xi;m) - |m|^{-\frac{n}{2}} e^{-i\frac{\lambda}{h}|m|} f(\frac{m}{|m|})]|_{M\setminus K_0} \in L^2$, or equivalently E_{inc} lies in the image of $C_0^\infty(\mathbb{R}^{n+1})$ under the free incoming resolvent $(h^2\Delta - (\lambda - i0)^2)^{-1}$ of the Laplacian on $\mathbb{R}^{n+1}$). These conditions provide a unique characterization of $E_h(\lambda,\xi)$. We can also write $E_h(\lambda,\xi) = E(\lambda/h,\xi)$, where $E(z,\xi)$ is the nonsemiclassical plane wave, and rewrite the results below in terms of the parameter z, as in the abstract.

We will freely use the notions of semiclassical analysis as found for example in [62], and reviewed in Section 3. We denote elements of the cotangent bundle T^*M by (m,ν), where $m \in M$ and $\nu \in T^*_m M$. The semiclassical principal symbol of $h^2\Delta$ is equal to $p(m,\nu) = |\nu|_g^2$, where $|\nu|_g$ is the length of $\nu \in T^*_m M$ with respect to the metric g. Therefore, the plane wave E_h should be concentrated on the unit cotangent bundle (see [62, Theorem 5.3])

$$S^*M := \{(m,\nu) \in T^*M \mid |\nu|_g = 1\}.$$

If $g^t : T^*M \to T^*M$ denotes the geodesic flow, then the Hamiltonian flow of p is $e^{tH_p} = g^{2t}$.

Semiclassical limits of E_h when the trapped set has measure zero. – In scattering theory trajectories which never escape to infinity play a special role as they can be observed only indirectly in asymptotics of plane waves. The *incoming tail* (resp. *outgoing tail*) $\Gamma_- \subset S^*M$ (resp. $\Gamma_+ \subset S^*M$) of the flow is defined as follows: a point (m,ν) lies in Γ_- (resp. Γ_+) if and only if the geodesic $g^t(m,\nu)$ stays in some compact set for $t \geq 0$ (resp. $t \leq 0$). The *trapped set* $K := \Gamma_+ \cap \Gamma_-$ is the set of points (m,ν) such that the geodesic $g^t(m,\nu)$ lies entirely in some compact subset of S^*M.

Our first result states that if $\mu_L(K) = 0$, then plane waves $E_h(\lambda,\xi)$ converge on average to some measures supported on the closure of the set of trajectories converging to ξ in $\overline{M}$:

THEOREM 1. – *Let (M,g) be a Riemannian manifold satisfying the assumptions of Section 4 and suppose that the trapped set has Liouville measure $\mu_L(K) = 0$. For Lebesgue almost every $\xi \in \partial\overline{M}$, there exists a Radon measure μ_ξ on S^*M such that for each compactly supported h-semiclassical pseudodifferential operator $A \in \Psi^0(M)$, we have as $h \to 0$,*

$$h^{-1}\left\| \langle AE_h(\lambda,\xi), E_h(\lambda,\xi)\rangle_{L^2(M)} - \int_{S^*M} \sigma(A)\, d\mu_\xi \right\|_{L^1_{\xi,\lambda}(\partial\overline{M}\times[1,1+h])} \to 0, \tag{1.5}$$

where $\sigma(A)$ is the semiclassical principal symbol of A as defined in [62, Theorem 14.1]. *The measure μ_ξ has support*

$$\operatorname{supp}(\mu_\xi) \subset \overline{\{(m,\nu) \in S^*M \mid \lim_{t\to+\infty} g^t(m,\nu) = \xi\}}, \tag{1.6}$$

and disintegrates the Liouville measure in the sense that there exists a smooth measure $d\xi$ on $\partial\overline{M}$ such that, if μ_L is the Liouville measure generated by $\sqrt{p} = |\nu|_g$ on S^*M, then

$$\int_{\partial\overline{M}} \mu_\xi \, d\xi = \mu_L. \tag{1.7}$$

The limiting measure μ_ξ is defined in Section 4.3. Implicit in (1.7) is the statement that for any bounded Borel $S \subset S^*M$, we have $\mu_\xi(S) \in L^1_\xi(\partial\overline{M})$. In Lemma A.1, we show that for hyperbolic manifolds μ_ξ is well defined for all $\xi \in \partial\overline{M}$ and it is likely that the same is true when the curvature of g is negative near the trapped set, but we believe that this does not hold in the general setting of Theorem 1.

In the case when $\mathrm{WF}_h(A) \cap \Gamma_- = \varnothing$ (in particular when g is non-trapping), we actually have a full expansion of $\langle AE_h, E_h\rangle$ in powers of h, with remainders bounded in $L^1_{\xi,\lambda}(\partial\overline{M} \times [1, 1+h])$—see (5.14). It is likely that for $K = \varnothing$, this can be strengthened to uniform convergence in ξ, λ, using nontrapping estimates on the resolvent.

The now standard argument of Colin de Verdière and Zelditch (see for example the proof of [62, Theorem 15.5]) shows that there exists a family of Borel sets $\mathcal{A}(h) \subset \partial\overline{M} \times [1, 1+h]$ such that the ratio of the measure of $\mathcal{A}(h)$ to the measure of the whole $\partial\overline{M} \times [1, 1+h]$ goes to 1 as $h \to 0$, and for each $A \in \Psi^0(M)$ as in Theorem 1 with $\sigma(A)$ independent of h,

$$\langle AE_h(\lambda,\xi), E_h(\lambda,\xi)\rangle_{L^2(M)} \to \int_{S^*M} \sigma(A)\, d\mu_\xi \text{ uniformly in } (\lambda,\xi) \in \mathcal{A}(h). \tag{1.8}$$

This statement can be viewed as an analogue of the quantum ergodicity fact (1.1), though as explained above, it is produced by a different phenomenon.

Estimates for the remainder. – We next provide a quantitative version of Theorem 1, namely an estimate of the left-hand side of (1.5). We define the set $\mathcal{T}(t)$ of geodesics trapped for time $t > 0$ as follows: let K_0 be a compact geodesically convex subset of M (in the sense of (B.1)) containing a neighborhood of the trapped set K, then (see also Section 5.2)

$$\mathcal{T}(t) := \{(m,\nu) \in S^*M \mid m \in K_0,\ \pi(g^t(m,\nu)) \in K_0\}, \tag{1.9}$$

where $\pi : T^*M \to M$ is the projection map. A quantity which will appear frequently with some parameter $\Lambda > 0$ is the following interpolated measure

$$r(h,\Lambda) := \sup_{0\le\theta\le1} h^{1-\theta} \mu_L\big(\mathcal{T}(\theta\Lambda^{-1}|\log h|)\big), \tag{1.10}$$

where $h > 0$ is small. This converges to 0 as $h \to 0$ when $\mu_L(K) = 0$ and it interpolates between h (when $\theta = 0$) and the Liouville measure of the set of geodesics that remain trapped for time $\Lambda^{-1}|\log h|$ (when $\theta = 1$). When the measure $\mu_L(\mathcal{T}(t))$ decays exponentially in t, as in (1.14), $r(h,\Lambda)$ can be replaced by simply $\mathcal{O}(h) + \mu_L(\mathcal{T}(\Lambda^{-1}|\log h|))$. The $\mathcal{O}(h)$ term here is natural since one can add an operator in $h\Psi^0(M)$ to A, which will change $\langle AE_h, E_h\rangle$ by $\mathcal{O}(h)$, but will not change $\sigma(A)$ (which is only defined invariantly modulo $\mathcal{O}(h)$).