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A THEOREM OF PALEY-WIENER TYPE FOR SCHRÖDINGER EVOLUTIONS

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ABSTRACT. – We prove unique continuation principles for solutions of evolution Schrödinger equations with time dependent potentials. These correspond to uncertainly principles of Paley-Wiener type for the Fourier transform. Our results extend to a large class of semi-linear Schrödinger equations.

RÉSUMÉ. – On prouve des principes de prolongement unique pour les solutions d'équations d'évolution de Schrödinger avec potentiels dépendant du temps. Ceux-ci correspondent à des principes d'incertitude de type Paley-Wiener pour la transformée de Fourier. Nos résultats se généralisent à une large classe d'équations de Schrödinger semi-linéaires.

1. Introduction

In this paper we study unique continuation properties of solutions of Schrödinger equations of the form

$$\partial_t u = i(\triangle u + V(x,t)u), \qquad (x,t) \in \mathbb{R}^n \times [0,T], \qquad T > 0. \tag{1.1}$$

The goal is to obtain sufficient conditions on the behavior of the solution u at two different times and on the potential V which guarantee that $u \equiv 0$ in $\mathbb{R}^n \times [0,T]$. Under appropriate assumptions this result will extend to the difference $v = u_1 - u_2$ of two solutions $u_1,\ u_2$ of semi-linear Schrödinger equation

$$\partial_t u = i(\triangle u + F(u,\overline{u})), \tag{1.2}$$

from which one can conclude that $u_1 \equiv u_2$.

Defining the Fourier transform of a function f as

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx,$$

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one has

$$u(x,t) = e^{it\Delta}u_0(x) = \int_{\mathbb{R}^n} \frac{e^{i|x-y|^2/4t}}{(4\pi it)^{n/2}}\, u_0(y)\, dy$$

$$= \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-2ix\cdot y/4t} e^{i|y|^2/4t} u_0(y)\, dy \tag{1.3}$$

$$= \frac{e^{i|x|^2/4t}}{(2it)^{n/2}}\ \widehat{(e^{i|\cdot|^2/4t}u_0)}\left(\frac{x}{2t}\right),$$

where $e^{it\Delta}u_0(x)$ denotes the free solution of the Schrödinger equation with data u_0

$$\partial_t u = i\triangle u, \qquad u(x,0) = u_0(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

The identity (1.3) tells us that this kind of results for the free solution of the Schrödinger equation are closely related to uncertainty principles for the Fourier transform. In this regard, one has the well known result of G. H. Hardy [9]:

$$\text{If } f(x) = O(e^{-x^2/\beta^2}),\ \ \widehat{f}(\xi) = O(e^{-4\,\xi^2/\alpha^2}) \text{ and } \alpha\beta < 4, \ \text{ then } f \equiv 0,$$
$$\text{and if } \alpha\beta = 4, \ \text{ then } f(x) = c\, e^{-x^2/\beta^2}.$$

Its extension to higher dimensions $n \geq 2$ was obtained in [15]. The following generalization in terms of the L^2-norm was established in [3]:

$$\text{If } \ e^{\frac{|x|^2}{\beta^2}}\, f(x),\ \ e^{\frac{4|\xi|^2}{\alpha^2}}\, \widehat{f}(\xi) \in L^2(\mathbb{R}^n), \text{ and } \alpha\,\beta \leq 4, \text{ then } f \equiv 0.$$

In terms of the free solution of the Schrödinger equation the L^2-version of Hardy Uncertainty Principle says :

$$\text{If } \ e^{\frac{|x|^2}{\beta^2}}\, u_0(x),\ \ e^{\frac{|x|^2}{\alpha^2}}\, e^{it\Delta}\, u_0(x) \in L^2(\mathbb{R}^n), \text{ and } \alpha\,\beta \leq 4t, \text{ then } u_0 \equiv 0. \tag{1.4}$$

In [6] the following result was proven:

THEOREM ([6]). – *Given any solution* $u \in C([0,T] : L^2(\mathbb{R}^n))$ *of*

$$\partial_t u = i\,(\triangle u + V(x,t)u)\,, \qquad (x,t) \in \mathbb{R}^n \times [0,T], \tag{1.5}$$

with $V \in L^\infty(\mathbb{R}^n \times [0,T])$,

$$\lim_{\rho\to+\infty} \|V\|_{L^1([0,T]:L^\infty(\mathbb{R}^n\setminus B_\rho))} = 0. \tag{1.6}$$

and

$$e^{\frac{|x|^2}{\beta^2}}\, u_0, \quad e^{\frac{|x|^2}{\alpha^2}}\, e^{iT\Delta}u_0 \in L^2(\mathbb{R}^n),$$

with $\alpha\,\beta < 4T$, *then* $u_0 \equiv 0$.

Notice that the above Theorem recovers the L^2-version of the Hardy Uncertainty Principle (1.4) for solutions of the IVP (1.5), except for the limiting case $\alpha\,\beta = 4T$ for which the corresponding result was proven to fail, see [6]. For further results in this direction concerning other uncertainty principles we refer to [8] and references therein.

Some previous results concerning uniqueness properties of solutions of the Schrödinger equation were not directly motivated by the Formula (1.3).

For solutions $u = u(x,t)$ of the 1-D cubic Schrödinger equation

$$\partial_t u = i(\partial_x^2 u \pm |u|^2 u), \tag{1.7}$$

B. Y. Zhang [17] showed :

If $u(x,t) = 0$ for $(x,t) \in (-\infty, a) \times \{0,1\}$ (or $(x,t) \in (a,\infty) \times \{0,1\}$) for some $a \in \mathbb{R}$, then $u \equiv 0$.

The proof is based on the inverse scattering method, which uses the fact that the equation in (1.7) is a completely integrable model.

In [13], under general assumptions on F in (1.2), it was proven that :

If $u_1,\ u_2 \in C([0,1] : H^s(\mathbb{R}^n))$, with $s > \max\{n/2;\ 2\}$ are solutions of the Equation (1.2) with F as in (1.2) such that

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Gamma_{x_0}^c \times \{0,1\},$$

where $\Gamma_{x_0}^c$ denotes the complement of a cone Γ_{x_0} with vertex $x_0 \in \mathbb{R}^n$ and opening $< 180^0$, then $u_1 \equiv u_2$.

For further results in this direction, see [12, 13], [10, 11] and references therein. Note that in [8] a unified approach was given to both kinds of results, using Lemma 3 and Corollary 1 below.

Returning to the uncertainty principle for the Fourier transform one has :

If $f \in L^1(\mathbb{R}^n)$ is non-zero and has compact support, then $\widehat{f}$ cannot satisfy a condition of the type $\widehat{f}(y) = O(e^{-\epsilon|y|})$ for any $\epsilon > 0$.

This is due to the fact that $\widehat{f}(y) = O(e^{-\epsilon|y|})$ implies that f has an analytic extension to the strip $\{z \in \mathbb{C}^n\ :\ |Im(z)| < \epsilon\}$.

In this regard the Paley-Wiener Theorem [14] gives a characterization of a function or distribution with compact support in term of the analyticity properties of its Fourier transform.

Our main result in this work is the following:

THEOREM 1. – *Let $u \in C([0,1] : L^2(\mathbb{R}^n))$ be a strong solution of the equation*

$$\partial_t u = i(\Delta u + V(x,t)u), \qquad (x,t) \in \mathbb{R}^n \times [0,1]. \tag{1.8}$$

Assume that

$$\sup_{0\le t\le 1} \int_{\mathbb{R}^n} |u(x,t)|^2 dx \le A_1, \tag{1.9}$$

$$\int_{\mathbb{R}^n} e^{2a_1|x_1|}\, |u(x,0)|^2\, dx = A_2 < \infty, \quad \textit{for some}\ \ a_1 > 0, \tag{1.10}$$

$$\operatorname{supp} u(\cdot,1) \subset \{x \in \mathbb{R}^n\ :\ x_1 \le a_2\}, \quad \textit{for some}\ \ a_2 < \infty, \tag{1.11}$$

with

$$V \in L^\infty(\mathbb{R}^n \times [0,1]), \qquad \|V\|_{L^\infty(\mathbb{R}^n\times[0,1])} = M_0, \tag{1.12}$$

and

$$\lim_{\rho\to+\infty} \|V\|_{L^1([0,1]:L^\infty(\mathbb{R}^n\setminus B_\rho))} = 0. \tag{1.13}$$

Then $u \equiv 0$.

REMARKS. – (a) Note that in order to prove Theorem 1, by translation in x_1, we can choose who a_2 is. We will show that there exists $m > 0$ (small) with the property that if (1.9), (1.10), (1.12), (1.13) hold and (1.11) holds with $a_2 = m$, then

$$u(x,1) = 0 \quad \text{for} \quad x \in \mathbb{R}^n \quad \text{such that} \quad m/2 < x_1 \le m.$$

This clearly yields the desired result. Without loss of generality we will assume $m < 1$.

(b) By rescaling it is clear that the result in Theorem 1 applies to any time interval $[0,T]$.

(c) We recall that in Theorem 1 there are no hypotheses on the size of the potential V in the given class or on its regularity.

(d) A weaker version of Theorem 1 was announced in [8].

As a direct consequence of Theorem 1 we get the following result regarding the uniqueness of solutions for non-linear equations of the form (1.2).

THEOREM 2. – *Given*

$$u_1,\, u_2 \in C([0,T] : H^k(\mathbb{R}^n)), \qquad 0 < T \le \infty,$$

strong solutions of (1.2) *with* $k \in \mathbb{Z}^+$, $k > n/2$, $F : \mathbb{C}^2 \to \mathbb{C}$, $F \in C^k$ *and* $F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0$ *such that*

$$\operatorname{supp}(u_1(\cdot,0) - u_2(\cdot,0)) \subset \{x \in \mathbb{R}^n : x_1 \le a_2\}, \qquad a_2 < \infty. \tag{1.14}$$

If for some $t \in (0,T)$ *and for some* $\epsilon > 0$

$$u_1(\cdot,t) - u_2(\cdot,t) \in L^2(e^{\epsilon|x_1|}\,dx), \tag{1.15}$$

then $u_1 \equiv u_2$.

REMARKS. – (a) In particular, by taking $u_2 \equiv 0$, Theorem 2 shows that if $u_1(\cdot,0)$ has compact support, then for any $t \in (0,T)$ $u_1(\cdot,t)$ cannot decay exponentially.

(b) In the case $F(u,\overline{u}) = |u|^{\alpha-1}u$, with $\alpha > n/2$ if α is not an odd integer, we have that if φ is the unique non-negative, radially symmetric solution of

$$-\Delta\varphi + \omega\,\varphi = |\varphi|^{\alpha-1}\varphi, \qquad \omega > 0,$$

then

$$u_1(x,t) = e^{i\omega t}\varphi(x) \tag{1.16}$$

is a solution ("standing wave") of

$$\partial_t u = i(\Delta u + |u|^{\alpha-1}u). \tag{1.17}$$

It was established in [16, 1] that there exist constants $c_0,\ c_1 > 0$ such that

$$\varphi(x) \le c_0\, e^{-c_1|x|}. \tag{1.18}$$