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SINGULAR CURVES AND THE ÉTALE BRAUER-MANIN OBSTRUCTION FOR SURFACES

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ABSTRACT. – We give an elementary construction of a smooth and projective surface over an arbitrary number field k that is a counterexample to the Hasse principle but has infinite étale Brauer-Manin set. Our surface has a surjective morphism to a curve with exactly one k-point such that the unique k-fibre is geometrically a union of projective lines with an adelic point and the trivial Brauer group, but no k-point.

RÉSUMÉ. – Nous présentons une construction élémentaire d'une surface lisse et projective sur un corps de nombres quelconque k qui constitue un contre-exemple au principe de Hasse et possède l'ensemble de Brauer-Manin infini. La surface est munie d'un morphisme surjectif vers une courbe avec un seul k-point tel que l'unique fibre rationnelle, qui géométriquement est l'union de droites projectives, a un point adélique et le groupe de Brauer trivial, mais pas de k-points.

Introduction

For a variety X over a number field k one can study the set $X(k)$ of k-points of X by embedding it into the topological space of adelic points $X(\mathbb{A}_k)$. In 1970 Manin [10] suggested to use the pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

provided by local class field theory. The left kernel of this pairing $X(\mathbb{A}_k)^{\mathrm{Br}}$ is a closed subset of $X(\mathbb{A}_k)$, and the reciprocity law of global class field theory implies that $X(k)$ is contained in $X(\mathbb{A}_k)^{\mathrm{Br}}$. The first example of a smooth and projective variety X such that $X(k) = \varnothing$ but $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$ was constructed in [18] (see [1] for a similar example; an earlier example conditional on the Bombieri-Lang conjecture was found in [14]). Later, Harari [6] found many varieties X such that $X(k)$ is not dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$. For all of these examples except for that of [14] the failure of the Hasse principle or weak approximation can be explained by the étale Brauer-Manin obstruction (introduced in [18], see also [13]): the closure of $X(k)$ in $X(\mathbb{A}_k)$ is contained in the étale Brauer-Manin set $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{Br}}$ which in these cases is smaller than $X(\mathbb{A}_k)^{\mathrm{Br}}$. Recently Poonen [13] constructed threefolds (fibred into rational surfaces over a curve of genus at least 1) such that $X(k) = \varnothing$ but $X(\mathbb{A}_k)^{\text{ét,Br}} \neq \varnothing$. It is known that $X(\mathbb{A}_k)^{\text{ét,Br}}$ coincides with the set of adelic points surviving the descent obstructions defined by torsors of arbitrary linear algebraic groups (as proved in [3, 17] using [7, 19]).

In 1997 Scharaschkin and the second author independently asked the question whether $X(k) = \varnothing$ if and only if $X(\mathbb{A}_k)^{\text{Br}} = \varnothing$ when X is a smooth and projective curve. They also asked if the embedding of $X(k)$ into $X(\mathbb{A}_k)^{\text{Br}}$ defines a bijection between the closure of $X(k)$ in $X(\mathbb{A}_k)$ and the set of connected components of $X(\mathbb{A}_k)^{\text{Br}}$. Despite some evidence for these conjectures, it may be prudent to consider also their weaker analogues with $X(\mathbb{A}_k)^{\text{ét,Br}}$ in place of $X(\mathbb{A}_k)^{\text{Br}}$.

In this note we give an elementary construction of a smooth and projective surface X over an arbitrary number field k that is a counterexample to the Hasse principle and has infinite étale Brauer-Manin set (Section 3). Even simpler is our counterexample to weak approximation (Section 2). This is a smooth and projective surface X over k with a unique k-point and infinite étale Brauer-Manin set $X(\mathbb{A}_k)^{\text{ét,Br}}$; moreover, infinitely many elements of $X(\mathbb{A}_k)^{\text{ét,Br}}$ have all their local components in the Zariski open set $X \setminus X(k)$. Following Poonen we consider families of curves parameterized by a curve with exactly one k-point. The new idea is to make the unique k-fibre a singular curve, geometrically a union of projective lines, and then use properties of rational and adelic points on singular curves.

The structure of the Picard group of a singular projective curve is well known, see [2, Section 9.2] or [9, Section 7.5]. In Section 1 we give a formula for the Brauer group of a reduced projective curve, see Theorem 1.3. A singular curve over k can have surprising properties that no smooth curve can ever have: it can contain infinitely many adelic points, only finitely many k-points or none at all, and yet have the trivial Brauer group. See Corollary 3.2 for a singular, geometrically connected, projective curve over an arbitrary number field k that is a counterexample to the Hasse principle not explained by the Brauer-Manin obstruction. In [8] the first author proves that every counterexample to the Hasse principle on a curve which geometrically is a union of projective lines, can be explained by finite descent (and hence by the étale Brauer-Manin obstruction). Here we note that geometrically connected and simply connected projective curves over number fields satisfy the Hasse principle, a statement that does not generalize to higher dimension, see Proposition 2.1 and Remark 2.2.

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1. The Brauer group of singular curves

Let k be a field of characteristic 0 with an algebraic closure $\bar{k}$ and the Galois group $\Gamma_k = \text{Gal}(\bar{k}/k)$. For a scheme X over k we write $\overline{X} = X \times_k \bar{k}$. All cohomology groups in this paper are Galois or étale cohomology groups. Let C be a *reduced, geometrically connected, projective* curve over k. We define the *normalization* $\widetilde{C}$ as the disjoint union of normalizations of the irreducible components of C. The normalization morphism $\nu : \widetilde{C} \to C$ factors as

$$\widetilde{C} \xrightarrow{\nu'} C' \xrightarrow{\nu''} C,$$

where C' is a maximal intermediate curve universally homeomorphic to C, see [2, Section 9.2, p. 247] or [9, Section 7.5, p. 308]. The curve C' is obtained from $\widetilde{C}$ by identifying the points which have the same image in C. In particular, there is a canonical bijection $\nu'' : C'(K)\tilde{\longrightarrow}C(K)$ for any field extension K/k. The curve C' has mildest possible singularities: for each singular point $s \in C'(\bar{k})$ the branches of $\overline{C}'$ through s intersect like n coordinate axes at $0 \in \mathbb{A}^n_{\bar{k}}$.

Let us define the following reduced 0-dimensional schemes:

$$\Lambda = \mathrm{Spec}(H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})), \ \ \Pi = C_{\mathrm{sing}}, \ \ \Psi = \big(\Pi \times_C \widetilde{C}\big)_{\mathrm{red}}. \tag{1.1}$$

Here Λ is the k-scheme of geometric irreducible components of C (or the geometric connected components of $\widetilde{C}$); it is the disjoint union of closed points $\lambda = \mathrm{Spec}(k(\lambda))$ such that $k(\lambda)$ is the algebraic closure of k in the function field of the corresponding irreducible component $k(C_\lambda) = k(\widetilde{C}_\lambda)$. Next, Π is the union of singular points of C, and Ψ is the union of fibres of $\nu : \widetilde{C} \to C$ over the singular points of C with their reduced subscheme structure. The morphism ν'' induces an isomorphism $\big(\Pi \times_C C'\big)_{\mathrm{red}}\tilde{\longrightarrow}\Pi$, so we can identify these schemes. Let $i : \Pi \to C$, $i' : \Pi \to C$ and $j : \Psi \to \widetilde{C}$ be the natural closed immersions. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\nu'} & C' & \xrightarrow{\nu''} & C \\ {\scriptstyle j}\uparrow & & {\scriptstyle i'}\uparrow & \nearrow{\scriptstyle i} & \\ \Psi & \xrightarrow{\nu'} & \Pi. & & \end{array}$$

The restriction of ν to the smooth locus of C induces isomorphisms

$$\widetilde{C} \setminus j(\Psi)\tilde{\longrightarrow}C' \setminus i'(\Pi)\tilde{\longrightarrow}C \setminus i(\Pi).$$

An algebraic group over Π is a product $G = \prod_\pi i_{\pi *}(G_\pi)$, where π ranges over the irreducible components of Π, $i_\pi : \mathrm{Spec}(k(\pi)) \to \Pi$ is the natural closed immersion, and G_π is an algebraic group over the field $k(\pi)$.

Proposition 1.1. – (i) *The canonical maps* $\mathbb{G}_{m,C'} \to \nu'_*\mathbb{G}_{m,\widetilde{C}}$ *and* $\mathbb{G}_{m,C'} \to i'_*\mathbb{G}_{m,\Pi}$ *give rise to the exact sequence of étale sheaves on* C'

$$0 \to \mathbb{G}_{m,C'} \to \nu'_*\mathbb{G}_{m,\widetilde{C}} \oplus i'_*\mathbb{G}_{m,\Pi} \to i'_*\nu'_*\mathbb{G}_{m,\Psi} \to 0, \tag{1.2}$$

where $\nu'_*\mathbb{G}_{m,\Psi}$ *is an algebraic torus over* Π.

(ii) *The canonical map* $\mathbb{G}_{m,C} \to \nu''_*\mathbb{G}_{m,C'}$ *gives rise to the exact sequence of étale sheaves on* C*:*

$$0 \to \mathbb{G}_{m,C} \to \nu''_*\mathbb{G}_{m,C'} \to i_*\mathcal{U} \to 0, \tag{1.3}$$

where $\mathcal{U}$ *is a commutative unipotent group over* Π.

Proof. – This is essentially well known, see [2], the proofs of Propositions 9.2.9 and 9.2.10, or [9, Lemma 7.5.12]. By [11, Thm. II.2.15 (b), (c)] it is enough to check the exactness of (1.2) at each geometric point $\bar{x}$ of C'. If $\bar{x} \notin i'(\Pi)$, this is obvious since locally at $\bar{x}$ the morphism ν' is an isomorphism, and the stalks $(i'_*\mathbb{G}_{m,\Pi})_{\bar{x}}$ and $(i'_*\nu'_*\mathbb{G}_{m,\Psi})_{\bar{x}}$ are zero. Now let $\bar{x} \in i'(\Pi)$, and let $\mathcal{O}_{\bar{x}}$ be the strict Henselisation of the local ring of $\bar{x}$ in C'. Each geometric point $\bar{y}$ of $\widetilde{C}$ belongs to exactly one geometric connected component of $\widetilde{C}$, and we denote by $\mathcal{O}_{\bar{y}}$

the strict Henselisation of the local ring of $\bar{y}$ in its geometric connected component. By the construction of C' we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{x}} \longrightarrow k(\bar{x}) \times \prod_{\nu'(\bar{y})=\bar{x}} \mathcal{O}_{\bar{y}} \longrightarrow \prod_{\nu'(\bar{y})=\bar{x}} k(\bar{y}) \longrightarrow 0,$$

where $\mathcal{O}_{\bar{y}} \to k(\bar{y})$ is the reduction modulo the maximal ideal of $\mathcal{O}_{\bar{y}}$, and $k(\bar{x}) \to k(\bar{y})$ is the multiplication by -1. We obtain an exact sequence of abelian groups

$$1 \longrightarrow \mathcal{O}^*_{\bar{x}} \longrightarrow k(\bar{x})^* \times \prod_{\nu'(\bar{y})=\bar{x}} \mathcal{O}^*_{\bar{y}} \longrightarrow \prod_{\nu'(\bar{y})=\bar{x}} k(\bar{y})^* \longrightarrow 1.$$

Using [11, Cor. II.3.5 (a), (c)] one sees that this is the sequence of stalks of (1.2) at $\bar{x}$, so that (i) is proved.

To prove (ii) consider the exact sequence

$$0 \to \mathbb{G}_{m,C} \to \nu''_*\mathbb{G}_{m,C'} \to \nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C} \to 0.$$

Since ν'' is an isomorphism away from $i(\Pi)$, the restriction of the sheaf $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C}$ to $C \setminus i(\Pi)$ is zero, hence $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C} = i_*\mathcal{U}$ for some sheaf $\mathcal{U}$ on Π. To see that $\mathcal{U}$ is a unipotent group scheme it is enough to check the stalks at geometric points. Let $\bar{x}$ be a geometric point of $i(\Pi)$, and let $\bar{y}$ be the unique geometric point of C' such that $\nu''(\bar{y}) = \bar{x}$. Let $\mathcal{O}_{\bar{x}}$ and $\mathcal{O}_{\bar{y}}$ be the corresponding strictly Henselian local rings. The stalk $(\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C})_{\bar{x}}$ is $\mathcal{O}^*_{\bar{y}}/\mathcal{O}^*_{\bar{x}}$, and according to [9, Lemma 7.5.12 (c)], this is a unipotent group over the field $k(\bar{x})$. This finishes the proof. □

REMARK 1.2. – The first part of Proposition 1.1 has an alternative proof which is easier to generalize to higher dimension. Let X be a projective k-variety with normalization morphism $\nu : \widetilde{X} \to X$. Assume that $\widetilde{X}$, X_{sing} and $\widetilde{X}_{\text{crit}}$ are smooth, where X_{sing} is the singular locus of X and $\widetilde{X}_{\text{crit}} = \nu^{-1}(X_{\text{sing}}) \subseteq \widetilde{X}$ is the critical locus of ν. (This assumption is automatically satisfied when X is a curve.) The analogue of C' is the K-variety X' given by the pushout in the square

$$\begin{array}{ccc} \widetilde{X}_{\text{crit}} & \xrightarrow{j} & \widetilde{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \nu'} \\ X_{\text{sing}} & \xrightarrow{i'} & X'. \end{array}$$

This pushout exists in the category of K-varieties since i' is a closed embedding and g is an affine morphism of smooth projective varieties (see [4, Thm. 5.4]). One then proves that the sequence of sheaves

$$0 \longrightarrow \mathbb{G}_{m,X'} \longrightarrow \nu'_*\mathbb{G}_{m,\widetilde{X}} \oplus i'_*\mathbb{G}_{m,X_{\text{sing}}} \longrightarrow \nu'_*j_*\mathbb{G}_{m,\widetilde{X}_{\text{crit}}} \longrightarrow 0$$

is exact, as follows. From the definition of X' we obtain that the square

$$\begin{array}{ccc} \mathcal{O}_{X'} & \longrightarrow & \nu'_*\mathcal{O}_{\widetilde{X}} \\ \downarrow & & \downarrow \\ i'_*\mathcal{O}_{X_{\text{sing}}} & \longrightarrow & \nu'_*j_*\mathcal{O}_{\widetilde{X}_{\text{crit}}} \end{array}$$