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COHOMOLOGY JUMP LOCI OF QUASI-PROJECTIVE VARIETIES

BY Nero BUDUR AND Botong WANG

ABSTRACT. – We prove that the cohomology jump loci in the space of rank one local systems over a smooth quasi-projective variety are finite unions of torsion translates of subtori. The main ingredients are a recent result of Dimca-Papadima, some techniques introduced by Simpson, together with properties of the moduli space of logarithmic connections constructed by Nitsure and Simpson.

RÉSUMÉ. – Dans cet article, on montre que les lieux de saut dans l'espace de systèmes locaux de rang un sur une variété lisse quasi-projective sont des réunions finies de subtores translatées par des éléments de torsion. Pour cela, nous utilisons un résultat récent de Dimca-Papadima, certaines techniques introduites par Simpson, ainsi que des propriétés de l'espace de moduli pour les connexions logarithmiques construit par Nitsure et Simpson.

1. Introduction

Let X be a connected, finite-type CW-complex. Define

$$\mathbf{M}_{\mathbf{B}}(X) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$$

to be the variety of $\mathbb{C}^*$ representations of $\pi_1(X)$. Then $\mathbf{M}_{\mathbf{B}}(X)$ is a direct product of $(\mathbb{C}^*)^{b_1(X)}$ and a finite Abelian group. For each point $\rho \in \mathbf{M}_{\mathbf{B}}(X)$, there exists a unique rank one local system L_ρ, whose monodromy representation is ρ. The *cohomology jump loci* of X are the natural strata

$$\Sigma^i_k(X) = \{\rho \in \mathbf{M}_{\mathbf{B}}(X) \mid \dim_{\mathbb{C}} H^i(X, L_\rho) \geq k\}.$$

$\Sigma^i_k(X)$ is a Zariski closed subset of $\mathbf{M}_{\mathbf{B}}(X)$. A celebrated result of Simpson says that if X is a smooth projective variety defined over $\mathbb{C}$, then $\Sigma^i_k(X)$ is a union of torsion translates of subtori of $\mathbf{M}_{\mathbf{B}}(X)$.

In this paper, we generalize Simpson's result to quasi-projective varieties.

THEOREM 1.1. – *Suppose U is a smooth quasi-projective variety defined over $\mathbb{C}$. Then $\Sigma^i_k(U)$ is a finite union of torsion translates of subtori of $\mathbf{M}_{\mathbf{B}}(U)$.*

When U is compact, the theorem is proved in [7, 8], [1], [15], with the strongest form appearing in the latter. When $b_1(\bar{U}) = 0$, Arapura [2] showed that $\Sigma^i_k(U)$ are union of translates of subtori. The case of unitary rank one local systems on U has been considered in [3]. Dimca and Papadima were able to prove the following:

THEOREM 1.2 ([6, Theorem C]). – *Under the same assumption as Theorem 1.1, every irreducible component of $\Sigma^i_k(U)$ containing $\mathbf{1} \in \mathbf{M}_\mathrm{B}(U)$ is a subtorus.*

The proof of this result reduces to the study of the infinitesimal deformations with cohomology constraints of the trivial local system. These are governed in general by infinite-dimensional models. In [6] it is shown that, in this case, the finite-dimensional Gysin model due to Morgan provides the necessary linear algebra description for the infinitesimal deformations.

The result of Dimca and Papadima serves as a key ingredient of our theorem. In Section 2, we will show that each irreducible component of $\Sigma^i_k(U)$ contains a torsion point. Then, in Section 3, we will see that, thanks to Theorem 1.2, having a torsion point on an irreducible component of $\Sigma^i_k(U)$ forces this component to be a translate of subtorus.

There are two other proofs of Simpson's theorem: one via positive characteristic methods [11], and one via D-modules [13, 12]. However, in this paper we follow the original approach of Simpson. There are no analogous results for higher rank local systems even in the projective case.

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2. Torsion points on the cohomology jump loci

Let X be a smooth complex projective variety, and let $D = \sum_{\lambda=1}^n D_\lambda$ be a simple normal crossing divisor on X with irreducible components D_λ. Let $U = X - D$. Thanks to Hironaka's theorem on resolution of singularities, every smooth quasi-projective variety U can be realized in this way. The goal of this section is to prove the following:

THEOREM 2.1. – *Each irreducible component of $\Sigma^i_k(U)$ contains a torsion point.*

First, we want to reduce to the case when X and each D_λ are defined over $\bar{\mathbb{Q}}$. This can be done using a technique which we have learnt from the proof of [15, Theorem 4.1]. We reproduce it here.

We can assume X and each D_λ to be defined over a subring O of $\mathbb{C}$, which is finitely generated over $\mathbb{Q}$. Denote the embedding of O to $\mathbb{C}$ by $\sigma : O \to \mathbb{C}$. Each ring homomorphism $O \to \mathbb{C}$ corresponds to a point in $\mathrm{Spec}(O)(\mathbb{C})$. Denote by X^0 and D^0_λ the schemes over $\mathrm{Spec}(O)$ which give rise to X and D_λ respectively after tensoring with $\mathbb{C}$, that is $X = X^0 \times_{\mathrm{Spec}(O)} \mathrm{Spec}(\mathbb{C})$ and $D_\lambda = D^0_\lambda \times_{\mathrm{Spec}(O)} \mathrm{Spec}(\mathbb{C})$. By possibly replacing O by $O[\frac{1}{h}]$ for some $h \in O$, we can assume X^0 and every D^0_λ are smooth over $\mathrm{Spec}(O)$, and all the intersections of D^0_λ's are transverse. Since each connected component of $\mathrm{Spec}(O)(\mathbb{C})$ contains a $\bar{\mathbb{Q}}$ point, there exists a point $P \in \mathrm{Spec}(O)(\bar{\mathbb{Q}})$, and a continuous path from $\sigma \in \mathrm{Spec}(O)(\mathbb{C})$ to P in $\mathrm{Spec}(O)(\mathbb{C})^{\mathrm{top}}$. Then, according to Thom's First Isotopy Lemma [5, Ch. 1, Theorem 3.5], $X^0(\mathbb{C})$ together with its strata given by the $D^0_\lambda(\mathbb{C})$, is a topologically locally trivial fibration in the stratified sense over $\mathrm{Spec}(O)(\mathbb{C})^{\mathrm{top}}$. In particular, letting X' and D'_λ be the corresponding fibers over P, transporting along the path gives an isomorphism $(X - D)^{\mathrm{top}} \cong (X' - D')^{\mathrm{top}}$. Recall that $\mathbf{M}_{\mathrm{B}}(U)$ and $\Sigma^i_k(U)$ depend only on the topology of U. Hence replacing $U = X - D$ by $U' = X' - D'$, we may assume that X and each D_λ are defined over $\bar{\mathbb{Q}}$.

Next, we introduce the other side of the story, namely the logarithmic flat bundles on (X, D). A logarithmic flat bundle on (X, D) consists of a vector bundle E on X, and a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$, satisfying the integrability condition $\nabla^2 = 0$. Given a logarithmic flat bundle (E, ∇), the flat sections of E on U (by which we will always mean on U^{top}) form a local system. And conversely, given any local system L on U (by which, as in the introduction, we will always mean a local system on U^{top}), it is always obtained from some logarithmic flat bundle (E, ∇). However, different logarithmic flat bundles may give the same local system. This correspondence between local systems on U and logarithmic flat bundles on (X, D) is very well understood (e.g., [4], [14], [9]).

For a vector bundle E on X, the structure of a logarithmic flat bundle (E, ∇) on (X, D) is the same as a $\mathcal{D}_X(\log D)$-module structure on E, where $\mathcal{D}_X(\log D)$ is the sheaf of logarithmic differentials.

Nitsure [10] and Simpson [16] constructed coarse moduli spaces, which are separated quasi-projective schemes, for Jordan-equivalence classes of semistable Λ-modules which are $\mathcal{O}_X$-coherent and torsion free, where Λ is a sheaf of rings of differential operators. The two examples of Λ which we are concerned with are $\mathcal{D}_X$, the usual sheaf of differential operators on X, and $\mathcal{D}_X(\log D)$, the sheaf of logarithmic differentials. We denote by $\mathbf{M}_{\mathrm{DR}}(X)$ and $\mathbf{M}_{\mathrm{DR}}(X/D)$ the moduli space of rank one $\mathcal{D}_X$-modules and the moduli space of rank one $\mathcal{D}_X(\log D)$-modules, respectively. In the rank one case, semistable is the same as stable and this condition is automatic as is the locally free condition, and Jordan-equivalence is the same as isomorphic. Thus, the points of $\mathbf{M}_{\mathrm{DR}}(X)$ and $\mathbf{M}_{\mathrm{DR}}(X/D)$ correspond to isomorphism classes of flat, respectively, logarithmic flat line bundles. Since we did not put any condition on the Chern class of the underlying line bundles, in general $\mathbf{M}_{\mathrm{DR}}(X/D)$ has infinitely many connected components. $\mathbf{M}_{\mathrm{DR}}(X)$, $\mathbf{M}_{\mathrm{DR}}(X/D)$, $\mathbf{M}_{\mathrm{B}}(X)$ and $\mathbf{M}_{\mathrm{B}}(U)$ are all algebraic groups, except $\mathbf{M}_{\mathrm{DR}}(X/D)$ may not be of finite type.

The diagram of Fig. 1 (p. 230) plays an essential role in our proof.

Let us first explain how the arrows are defined. Since every $\mathcal{D}_X$-module is naturally a $\mathcal{D}_X(\log D)$-module, there is a natural embedding $\mathbf{M}_{\mathrm{DR}}(X) \hookrightarrow \mathbf{M}_{\mathrm{DR}}(X/D)$. On the other hand, the embedding $U \hookrightarrow X$ induces a surjective map on the fundamental group $\pi_1(U) \to \pi_1(X)$. Composing this map with the representations, we have $\mathbf{M}_{\mathrm{B}}(X) \hookrightarrow \mathbf{M}_{\mathrm{B}}(U)$. For every rank one logarithmic flat bundle (E, ∇), taking the residue along each D_λ is the map res. In other words, $\mathrm{res}((E, \nabla)) = \{\mathrm{res}_{D_\lambda}(\nabla)\}_{1 \le \lambda \le n}$. Around each D_λ, we can take a small loop γ_λ. The map ev is the evaluation at the loops γ_λ. More precisely $\mathrm{ev}(\rho) = \{\rho(\gamma_\lambda)\}_{1 \le \lambda \le n}$.

For the horizontal arrows, $RH : \mathbf{M}_{\mathrm{DR}}(X) \to \mathbf{M}_{\mathrm{B}}(X)$ is taking the monodromy representations for flat bundles. Since every logarithmic flat bundle on (X, D) restricts to a flat bundle on U, taking the monodromy representation on U is $RH : \mathbf{M}_{\mathrm{DR}}(X/D) \to \mathbf{M}_{\mathrm{B}}(U)$. The map $\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is component-wise defined to be multiplying by $2\pi\sqrt{-1}$, then taking exponential. On $\mathbf{M}_{\mathrm{DR}}(X/D)$, there are some special elements. Let $(\mathcal{O}_X, d)$ be the