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*Deformations of Levi flat hypersurfaces in complex manifolds*

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## DEFORMATIONS OF LEVI FLAT HYPERSURFACES IN COMPLEX MANIFOLDS

BY PAOLO DE BARTOLOMEIS AND ANDREI IORDAN

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**ABSTRACT.** – We first give a deformation theory of integrable distributions of codimension 1. This theory is used to study Levi-flat deformations: a Levi-flat deformation of a Levi flat hypersurface  $L$  in a complex manifold is a smooth mapping  $\Psi : I \times M \rightarrow M$  such that  $\Psi_t = \Psi(t, \cdot) \in \text{Diff}(M)$ ,  $L_t = \Psi_t L$  is a Levi flat hypersurface in  $M$  for every  $t \in I$  and  $L_0 = L$ . We define a parametrization of families of smooth hypersurfaces near  $L$  such that the Levi flat deformations are given by the solutions of the Maurer-Cartan equation in a DGLA associated to the Levi foliation. We say that  $L$  is infinitesimally rigid if the tangent cone at the origin to the moduli space of Levi flat deformations of  $L$  is trivial. We prove the infinitesimal rigidity of compact transversally parallelizable Levi flat hypersurfaces in compact complex manifolds and give sufficient conditions for infinitesimal rigidity in Kähler manifolds. As an application, we prove the nonexistence of transversally parallelizable Levi flat hypersurfaces in a class of manifolds which contains  $\mathbb{C}P_2$ .

**RÉSUMÉ.** – Nous commençons par présenter une théorie des déformations de distributions intégrables de codimension 1. Cette théorie est utilisée pour étudier les déformations d’hypersurfaces Levi plates: une déformation Levi plate d’une hypersurface Levi plate  $L$  dans une variété complexe  $M$  est une application lisse  $\Psi : I \times M \rightarrow M$  telle que  $\Psi_t = \Psi(t, \cdot) \in \text{Diff}(M)$ ,  $L_t = \Psi_t L$  est une hypersurface Levi plate dans  $M$  pour tout  $t \in I$  et  $L_0 = L$ . Nous définissons une paramétrisation des hypersurfaces Levi plates au voisinage de  $L$  telle que les déformations d’hypersurfaces Levi plates de  $L$  sont données par les solutions de l’équation de Maurer-Cartan dans une DGLA associée au feuilletage de Levi.

Nous disons que  $L$  est infinitésimalement rigide si le cône tangent à l’origine de l’espace de modules des déformations Levi plates de  $L$  est trivial. Nous prouvons que les hypersurfaces de Levi plates compactes transversalement parallélisables dans les variétés complexes compactes sont infinitésimalement rigides et nous donnons des conditions suffisantes pour la rigidité infinitésimale dans les variétés de Kähler. Comme application, nous démontrons la non existence d’hypersurfaces Levi plates transversalement parallélisables dans une classe de variétés qui contient l’espace projectif complexe de dimension  $n \geq 2$ .

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## 1. Introduction

Let  $M$  be a complex manifold and  $L$  a real hypersurface of class  $C^2$  in  $M$  such that  $M \setminus L = \Omega_1 \cup \Omega_2$ ,  $\Omega_1 \cap \Omega_2 = \emptyset$ .  $L$  is Levi flat if it satisfies one of the following equivalent conditions:

- 1)  $\Omega_1$  and  $\Omega_2$  are pseudoconvex domains.
- 2)  $L$  is foliated by complex hypersurfaces of  $M$ .
- 3) The Levi form of  $L$  vanishes.

It is well known that in general, if  $L$  is not of class  $C^2$ , we have only 3)  $\implies$  2)  $\implies$  1).

One of the oldest result concerning Levi flat hypersurfaces is a theorem of E. Cartan [3] which states that a real analytic Levi flat hypersurface is locally isomorphic to the set of vanishing of the real part of a holomorphic function. A generalization of this theorem for singular Levi flat hypersurfaces can be found in [9].

Recent research on Levi flat hypersurfaces in complex manifolds were motivated by the following conjecture of D. Cerveau [4]: there are no smooth Levi flat hypersurfaces in the complex projective space  $\mathbb{C}\mathbb{P}_n$ ,  $n \geq 2$ .

For  $n \geq 3$ , this conjecture was proved by Lins Neto for real analytic Levi flat hypersurfaces [15], by Y.-T. Siu for Levi flat hypersurfaces of class  $C^{1,2}$  [18] and by A. Iordan and F. Matthey for Lipschitz hypersurfaces of Sobolev class  $W^s$ ,  $s > 5/2$  [11]. Despite several attempts to prove this conjecture for  $n = 2$ , its proof is still incomplete.

Unlike  $\mathbb{C}\mathbb{P}_n$ ,  $n \geq 2$ , the complex tori  $\mathbb{T}_n = \mathbb{C}^n/\Gamma$  contains the Levi flat hypersurfaces  $\pi(\oplus_{j=1}^{2n-1} \mathbb{R}u_j + u)$  where  $\pi : \mathbb{C}^n \rightarrow \mathbb{T}_n$  is the canonical projection,  $u_j$ ,  $j = 1, \dots, 2n-1$ , are  $\mathbb{R}$ -linearly independent vectors in  $\Gamma$  and  $u \in \mathbb{C}^n$  [16]. It was conjectured in [16] that for every compact Levi flat hypersurface  $M$  in  $\mathbb{T}_n$ ,  $\pi^{-1}(M)$  is a union of affine hyperplanes.

In this paper we study the deformations of smooth Levi flat hypersurfaces in complex manifolds. The theory of deformations of complex manifolds was intensively studied from the 50s beginning with the famous results of Kodaira and Spencer [13] (see for ex. [12], [21]). In [17], Nijenhuis and Richardson adapted a theory initiated by Gerstenhaber [6] and proved the connection between the deformations of complex analytic structures and the theory of differential graded Lie algebras (DGLA). This theory was developed following ideas of Deligne by Goldman and Millson [8].

The main results of this paper may be summarized as follows.

In the first chapter we consider integrable distributions of codimension 1 on smooth manifolds and we define a DGLA associated to the foliation such that the deformations of integrable distributions of codimension 1 are given by solutions of Maurer-Cartan equation in this algebra. As the examples show, this theory is highly non trivial and it seems to be interesting by itself. We mention that Kodaira and Spencer developed in [14] a theory of deformations of the so called multifoliate structures, which are more general than the foliate structures. Our approach in this paper for foliations of codimension 1 is different of theirs (see Remark 14) and allows us to study the Levi flat case.

In the second chapter we give a description of the deformations of a smooth Levi flat hypersurface  $L$  in a complex manifold by means of the Maurer-Cartan equation in the DGLA associated to the Levi foliation.

Then we establish the equations verified by the tangent to a regular family of Levi flat deformations. We say that  $L$  is infinitesimally rigid (respectively strongly infinitesimally rigid) if the tangent cone at the origin to the moduli space of Levi flat deformations of  $L$  is trivial (respectively if the tangent cone at the origin to the solutions of the Maurer-Cartan equation in the DGLA associated to the Levi foliation is trivial). We remark that Diederich and Ohsawa study in [5] the displacement rigidity of Levi flat hypersurfaces in disc bundle over compact Riemann surfaces. The definition of rigidity in [5] means that any small  $C^2$  perturbation of a Levi flat hypersurface  $L$  is CR isomorphic with  $L$ , so  $L$  is strongly infinitesimally rigid.

We prove that a transversally parallelizable compact Levi flat hypersurface in a compact complex manifold is strongly infinitesimally rigid and we give a sufficient condition for infinitesimal rigidity in Kähler manifolds (Theorem 3). As an application, we prove that there are no compact transversally parallelizable Levi flat hypersurfaces in connected complex manifolds  $M$  such that for every  $p \neq q \in M$  and every real hyperplane  $H_q$  in  $T_qM$  there exists a holomorphic vector field  $Y$  on  $M$  such that  $Y(p) = 0$  and  $Y(q) \oplus H_q = T_qM$ . If  $M = \mathbb{C}\mathbb{P}_n, n \geq 2$ , the hypotheses of the previous result are fulfilled.

The non existence of transversally parallelizable Levi flat hypersurfaces in  $\mathbb{C}\mathbb{P}_2$  can be obtained by different proofs. We chose here to give a proof by using the results of this paper. Another direct proof was furnished to the authors by Marco Brunella [2] who disappeared recently in a tragic accident. We want to pay tribute to the memory of Marco Brunella by giving also his proof of this result.

## 2. Deformation theory of integrable distribution of codimension 1

### 2.1. DGLA associated to an integrable distribution of codimension 1

DEFINITION 1. – A differential graded Lie algebra (DGLA) is a triple  $(V^*, d, [\cdot, \cdot])$  such that:

1)  $V^* = \bigoplus_{i \in \mathbb{N}} V^i$ , where  $(V^i)_{i \in \mathbb{N}}$  is a family of  $\mathbb{C}$ -vector spaces and  $d : V^* \rightarrow V^*$  is a graded homomorphism such that  $d^2 = 0$ . An element  $a \in V^k$  is said to be homogeneous of degree  $k = \deg a$ .

2)  $[\cdot, \cdot] : V^* \times V^* \rightarrow V^*$  defines a structure of graded Lie algebra i.e., for homogeneous elements we have

$$(2.1) \quad [a, b] = -(-1)^{\deg a \deg b} [b, a]$$

and

$$(2.2) \quad [a, [b, c]] = [[a, b], c] + (-1)^{\deg a \deg b} [b, [a, c]].$$

3)  $d$  is compatible with the graded Lie algebra structure i.e.,

$$(2.3) \quad d[a, b] = [da, b] + (-1)^{\deg a} [a, db].$$

REMARK 1. – If (2.1) is satisfied then (2.2) is equivalent to

$$(2.4) \quad \mathfrak{S}_s (-1)^{\deg a \deg c} [a, [b, c]] = 0$$

where  $\mathfrak{S}_s$  denotes the symmetric sum.