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GEOMETRY AND TOPOLOGY OF COMPLETE LORENTZ SPACETIMES OF CONSTANT CURVATURE

BY Jeffrey DANCIGER, François GUÉRITAUD
AND Fanny KASSEL

Abstract. – We study proper, isometric actions of non virtually solvable discrete groups Γ on the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$, viewing them as limits of actions on the 3-dimensional anti-de Sitter space AdS^3. To each such action on $\mathbb{R}^{2,1}$ is associated an infinitesimal deformation, inside $\mathrm{SO}(2,1)$, of the fundamental group of a hyperbolic surface S. When S is convex cocompact, we prove that Γ acts properly on $\mathbb{R}^{2,1}$ if and only if this group-level deformation is realized by a deformation of S that uniformly contracts or uniformly expands all distances. We give two applications in this case. (1) Tameness: A complete flat spacetime is homeomorphic to the interior of a compact manifold with boundary. (2) Geometric transition: A complete flat spacetime is the rescaled limit of collapsing AdS spacetimes.

Résumé. – Nous étudions les actions propres, par isométries, de groupes discrets non virtuellement résolubles Γ sur l'espace de Minkowski $\mathbb{R}^{2,1}$, en les voyant comme limites d'actions sur l'espace anti-de Sitter AdS^3. À une telle action sur $\mathbb{R}^{2,1}$ est associée une déformation infinitésimale, dans $\mathrm{SO}(2,1)$, du groupe fondamental d'une surface hyperbolique S. Lorsque S est convexe cocompacte, nous montrons que Γ agit proprement sur $\mathbb{R}^{2,1}$ si et seulement si cette déformation au niveau du groupe est réalisée par une déformation de S qui contracte uniformément ou dilate uniformément toutes les distances. Nous donnons deux applications dans ce cas. (1) Sagesse topologique : un espace-temps plat complet est homéomorphe à l'intérieur d'une variété compacte à bord. (2) Transition géométrique : un espace-temps plat complet est la limite renormalisée d'espaces-temps AdS qui dégénèrent.

1. Introduction

A Lorentzian 3-manifold of constant negative curvature is locally modeled on the *anti-de Sitter* space $\mathrm{AdS}^3 = \mathrm{PO}(2,2)/\mathrm{O}(2,1)$, which can be realized in $\mathbb{RP}^3$ as the set of negative points with respect to a quadratic form of signature $(2,2)$. A flat Lorentzian 3-manifold is locally modeled on the *Minkowski* space $\mathbb{R}^{2,1}$, which is the affine space $\mathbb{R}^3$ endowed with

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the Lorentzian structure induced by a quadratic form of signature $(2,1)$. Observe that the tangent space at a point of AdS^3 identifies with $\mathbb{R}^{2,1}$; this basic fact motivates the point of view of this paper that a large class of manifolds modeled on $\mathbb{R}^{2,1}$ (convex cocompact Margulis spacetimes) are infinitesimal versions of manifolds modeled on AdS^3. We consider only *complete* Lorentzian manifolds which are quotients of AdS^3 or $\mathbb{R}^{2,1}$ by discrete groups Γ of isometries acting properly discontinuously.

The following facts, specific to dimension 3, will be used throughout the paper. The anti-de Sitter space AdS^3 identifies with the manifold $G = \mathrm{PSL}_2(\mathbb{R})$ endowed with the Lorentzian metric induced by (a multiple of) the Killing form. The group of orientation and time-orientation preserving isometries is $G \times G$ acting by right and left multiplication: $(g_1, g_2)\cdot g = g_2 g g_1^{-1}$. The Minkowski space $\mathbb{R}^{2,1}$ can be realized as the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. The group of orientation and time-orientation preserving isometries is $G \ltimes \mathfrak{g}$ acting affinely: $(g, v) \cdot w = \mathrm{Ad}(g)w + v$.

Examples of groups of isometries acting properly discontinuously on AdS^3 are easy to construct: one can take $\Gamma = \Gamma_0 \times \{1\}$ where Γ_0 is any discrete subgroup of G; in this case the quotient $\Gamma\backslash\mathrm{AdS}^3$ identifies with the unit tangent bundle to the hyperbolic orbifold $\Gamma_0\backslash\mathbb{H}^2$. Such quotients are called standard. Goldman [18] produced the first nonstandard examples by deforming standard ones, a technique that was later generalized by Kobayashi [30]. Salein [38] constructed the first examples that were not deformations of standard ones.

On the other hand, although cyclic examples are readily constructed, it is not obvious that there exist *nonsolvable* groups acting properly discontinuously on $\mathbb{R}^{2,1}$. The Auslander conjecture in dimension 3, proved by Fried-Goldman [17], states that any discrete group acting properly discontinuously *and cocompactly* on $\mathbb{R}^{2,1}$ is solvable up to finite index, generalizing Bieberbach's theory of crystallographic groups. Milnor [36] asked if the cocompactness assumption could be removed. This was answered negatively by Margulis [33, 34], who constructed the first examples of nonabelian free groups acting properly discontinuously on $\mathbb{R}^{2,1}$ (see [14] for another proof); the quotient manifolds coming from such actions are now often called *Margulis spacetimes*. Drumm [12, 13] constructed more examples of Margulis spacetimes by introducing polyhedral surfaces called *crooked planes* to produce fundamental domains.

1.1. Proper actions and contraction

A discrete group Γ acting on AdS^3 by isometries that preserve both orientation and time orientation is determined by two representations $j, \rho : \Gamma \to G = \mathrm{PSL}_2(\mathbb{R})$, the *first projection* and *second projection*. We refer to the group of isometries determined by (j, ρ) using the notation $\Gamma^{j,\rho}$. By work of Kulkarni-Raymond [31], if such a group $\Gamma^{j,\rho}$ acts properly on AdS^3 and is torsion-free, then one of the representations j, ρ must be injective and discrete; if Γ is finitely generated (which we shall always assume), then we may pass to a finite-index subgroup that is torsion-free by the Selberg lemma [40, Lem. 8]. We assume then that j is injective and discrete. When j is convex cocompact, Kassel [27] gave a full characterization of properness of the action of $\Gamma^{j,\rho}$ in terms of a double contraction condition. Specifically, $\Gamma^{j,\rho}$ acts properly on AdS^3 if and only if either of the following two equivalent conditions holds (up to switching j and ρ if both are convex cocompact):

- (*Lipschitz contraction*) There exists a (j,ρ)-equivariant Lipschitz map $f : \mathbb{H}^2 \to \mathbb{H}^2$ with Lipschitz constant < 1;
- (*Length contraction*)

$$\sup_{\gamma\in\Gamma \text{ with } \lambda(j(\gamma))>0} \frac{\lambda(\rho(\gamma))}{\lambda(j(\gamma))} < 1, \tag{1.1}$$

where $\lambda(g)$ is the hyperbolic translation length of $g \in G$ (defined to be 0 if g is not hyperbolic, see (2.1)). This was extended by Guéritaud-Kassel [23] to the case that the finitely generated group $j(\Gamma)$ is allowed to have parabolic elements. The two (equivalent) types of contraction appearing above are easy to illustrate in the case when ρ is also discrete and injective: the Lipschitz contraction criterion says that there exists a map $j(\Gamma)\backslash\mathbb{H}^2 \to \rho(\Gamma)\backslash\mathbb{H}^2$ (in the correct homotopy class) that uniformly contracts all distances on the surface, while the length contraction criterion says that any closed geodesic on $\rho(\Gamma)\backslash\mathbb{H}^2$ is uniformly shorter than the corresponding geodesic on $j(\Gamma)\backslash\mathbb{H}^2$. Lipschitz contraction easily implies length contraction, but the converse is not obvious. One important consequence that can be deduced from either criterion is that for a fixed convex cocompact j, the representations ρ that yield a proper action form an open set. In Section 6 (which can be read independently), we derive topological and geometric information about the quotient manifold directly from the Lipschitz contraction property.

We remark that $\Gamma^{j,\rho}$ does not act properly on AdS^3 in the case that Γ is a closed surface group and j,ρ are both Fuchsian (i.e., injective and discrete). For Thurston showed, as part of his theory of the asymmetric metric on Teichmüller space [42], that the best Lipschitz constant of maps $j(\Gamma)\backslash\mathbb{H}^2 \to \rho(\Gamma)\backslash\mathbb{H}^2$ (in the correct homotopy class) is ≥ 1, with equality only if ρ is conjugate to j. However, $\Gamma^{j,\rho}$ does act properly on a convex subdomain of AdS^3; the resulting AdS manifolds are the globally hyperbolic spacetimes studied by Mess [35].

We now turn to the flat case. A discrete group Γ acting on $\mathbb{R}^{2,1}$ by isometries that preserve both orientation and time orientation is determined by a representation $j : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ and a j-cocycle $u : \Gamma \to \mathfrak{sl}_2(\mathbb{R})$, i.e., a map satisfying

$$u(\gamma_1\gamma_2) = u(\gamma_1) + \mathrm{Ad}(j(\gamma_1))\, u(\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. We refer to the group of isometries determined by (j,u) using the notation $\Gamma^{j,u}$, where j gives the *linear part* and u the *translational part* of $\Gamma^{j,u}$. The cocycle u may be thought of as an infinitesimal deformation of j (see Section 2.3). Fried-Goldman [17] showed that if Γ acts properly on $\mathbb{R}^{2,1}$ and is not virtually solvable, then j must be injective and discrete on a finite-index subgroup of Γ; in particular $j(\Gamma)$ is the fundamental group of a hyperbolic surface S (up to finite index). Unlike in the AdS case, here S cannot be compact (see Mess [35]). In the case that it is convex cocompact, Goldman-Labourie-Margulis [20] gave a properness criterion in terms of the so-called *Margulis invariant*. Given the interpretation of this invariant as a derivative of translation lengths [22], the group $\Gamma^{j,u}$ (with j convex cocompact) acts properly on $\mathbb{R}^{2,1}$ if and only if, up to replacing u by $-u$, the infinitesimal deformation u contracts the lengths of all closed geodesics on S at a uniform rate:

$$\sup_{\gamma\in\Gamma \text{ with } \lambda(j(\gamma))>0} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\lambda(e^{tu(\gamma)}j(\gamma))}{\lambda(j(\gamma))} < 0\,. \tag{1.2}$$