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# ON SEMI-CLASSICAL LIMIT OF NONLINEAR QUANTUM SCATTERING

BY Rémi CARLES

Abstract. – We consider the nonlinear Schrödinger equation with a short-range external potential, in a semi-classical scaling. We show that for fixed Planck constant, a complete scattering theory is available, showing that both the potential and the nonlinearity are asymptotically negligible for large time. Then, for data under the form of coherent state, we show that a scattering theory is also available for the approximate envelope of the propagated coherent state, which is given by a nonlinear equation. In the semi-classical limit, these two scattering operators can be compared in terms of classical scattering theory, thanks to a uniform in time error estimate. Finally, we infer a large time decoupling phenomenon in the case of finitely many initial coherent states.

Résumé. – Nous considérons l'équation de Schrödinger non linéaire en présence d'un potentiel à courte portée, en régime semi-classique. Lorsque la constante de Planck est fixée, une théorie du scattering permet d'établir qu'à la fois le potentiel et la non-linéarité sont négligeables en temps grand. Par ailleurs, pour des données sous la forme d'états cohérents, nous établissons une théorie du scattering pour l'équation d'enveloppe, elle-même non linéaire. Dans la limite semi-classique, les deux opérateurs de scattering peuvent être comparés, en faisant intervenir en outre la théorie du scattering classique, grâce à une estimation d'erreur uniforme en temps. Enfin, nous déduisons un phénomène de découplage en temps grand dans le cas d'un nombre fini d'états cohérents.

## 1. Introduction

We consider the equation

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + |\psi^\varepsilon|^2\psi^\varepsilon, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3, \tag{1.1}$$

and both semi-classical ($\varepsilon \to 0$) and large time ($t \to \pm\infty$) limits. Of course these limits must not be expected to commute, and one of the goals of this paper is to analyze this lack of commutation on specific asymptotic data, under the form of coherent states as described

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below. Even though our main result (Theorem 1.7) is proven specifically for the above case of a cubic three-dimensional equation, two important intermediate results (Theorems 1.4 and 1.5) are established in a more general setting. Unless specified otherwise, we shall from now on consider $\psi^\varepsilon : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$, $d \geqslant 1$.

### 1.1. Propagation of initial coherent states

In this subsection, we consider the initial value problem, as opposed to the scattering problem treated throughout this paper. More precisely, we assume here that the wave function is, at time $t = 0$, given by the coherent state

$$\psi^\varepsilon(0,x) = \frac{1}{\varepsilon^{d/4}} a\left(\frac{x-q_0}{\sqrt{\varepsilon}}\right) e^{ip_0\cdot(x-q_0)/\varepsilon}, \tag{1.2}$$

where $q_0, p_0 \in \mathbb{R}^d$ denote the initial position and velocity, respectively. The function $a$ belongs to the Schwartz class, typically. In the case where $a$ is a (complex) Gaussian, many explicit computations are available in the linear case (see [33]). Note that the $L^2$-norm of $\psi^\varepsilon$ is independent of $\varepsilon$, $\|\psi^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|a\|_{L^2(\mathbb{R}^d)}$. Coherent states are of particular importance in quantum mechanics. Several definitions are available, involving several fields in mathematics: Lie groups and complex analysis, functional analysis (quantization), partial differential equations, for instance (see [17] and references therein). In the latter case, they are usually associated with the linear Schrödinger equation.

Throughout this subsection, we assume that the external potential $V$ is smooth and real-valued, $V \in C^\infty(\mathbb{R}^d;\mathbb{R})$, and at most quadratic, in the sense that

$$\partial^\alpha V \in L^\infty(\mathbb{R}^d), \quad \forall |\alpha| \geqslant 2.$$

This assumption will be strengthened when large time behavior is analyzed.

1.1.1. *Linear case.* – Resume (1.1) in the absence of nonlinear term:

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon, \quad x \in \mathbb{R}^d, \tag{1.3}$$

associated with the initial datum (1.2). To derive an approximate solution, and to describe the propagation of the initial wave packet, introduce the Hamiltonian flow

$$\dot{q}(t) = p(t), \quad \dot{p}(t) = -\nabla V\left(q(t)\right), \tag{1.4}$$

and prescribe the initial data $q(0) = q_0$, $p(0) = p_0$. Since the potential $V$ is smooth and at most quadratic, the solution $(q(t), p(t))$ is smooth, defined for all time, and grows at most exponentially. The classical action is given by

$$S(t) = \int_0^t \left(\frac{1}{2}|p(s)|^2 - V(q(s))\right) ds. \tag{1.5}$$

We observe that if we change the unknown function $\psi^\varepsilon$ to $u^\varepsilon$ by

$$\psi^\varepsilon(t,x) = \varepsilon^{-d/4} u^\varepsilon\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+p(t)\cdot(x-q(t)))/\varepsilon}, \tag{1.6}$$

then, in terms of $u^\varepsilon = u^\varepsilon(t,y)$, the Cauchy problem (1.3)–(1.2) is equivalent to

$$i\partial_t u^\varepsilon + \frac{1}{2}\Delta u^\varepsilon = V^\varepsilon(t,y)u^\varepsilon; \quad u^\varepsilon(0,y) = a(y), \tag{1.7}$$

where the external time-dependent potential $V^\varepsilon$ is given by

$$V^\varepsilon(t,y) = \frac{1}{\varepsilon}\left(V(q(t)+\sqrt{\varepsilon}y) - V(q(t)) - \sqrt{\varepsilon}\left\langle \nabla V(q(t)), y\right\rangle\right). \tag{1.8}$$

This potential corresponds to the first term of a Taylor expansion of $V$ about the point $q(t)$, and we naturally introduce $u = u(t,y)$ solution to

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\left\langle Q(t)y,y\right\rangle u\,;\quad u(0,y) = a(y), \tag{1.9}$$

where

$$Q(t) := \nabla^2 V\left(q(t)\right),\quad \text{so that } \frac{1}{2}\left\langle Q(t)y,y\right\rangle = \lim_{\varepsilon\to 0} V^\varepsilon(t,y).$$

The obvious candidate to approximate the initial wave function $\psi^\varepsilon$ is then:

$$\varphi^\varepsilon(t,x) = \varepsilon^{-d/4} u\left(t, \frac{x-q(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t)+p(t)\cdot(x-q(t)))/\varepsilon}. \tag{1.10}$$

Indeed, it can be proven (see e.g., [2, 4, 17, 33, 35, 36]) that there exists $C>0$ independent of $\varepsilon$ such that

$$\|\psi^\varepsilon(t,\cdot) - \varphi^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leqslant C\sqrt{\varepsilon}e^{Ct}.$$

Therefore, $\varphi^\varepsilon$ is a good approximation of $\psi^\varepsilon$ at least up to time of order $c\ln\frac{1}{\varepsilon}$ (Ehrenfest time).

1.1.2. *Nonlinear case.* – When adding a nonlinear term to (1.3), one has to be cautious about the size of the solution, which rules the importance of the nonlinear term. To simplify the discussions, we restrict our analysis to the case of a gauge invariant, defocusing, power nonlinearity, $|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon$. We choose to measure the importance of nonlinear effects not directly through the size of the initial data, but through an $\varepsilon$-dependent coupling factor: we keep the initial datum (1.2) (with an $L^2$-norm independent of $\varepsilon$), and consider

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon.$$

Since the nonlinearity is homogeneous, this approach is equivalent to considering $\alpha=0$, up to multiplying the initial datum by $\varepsilon^{\alpha/(2\sigma)}$. We assume $\sigma>0$, with $\sigma<2/(d-2)$ if $d\geqslant 3$: for $a\in\Sigma$, defined by

$$\Sigma = \{f\in H^1(\mathbb{R}^d),\quad x\mapsto \langle x\rangle f(x)\in L^2(\mathbb{R}^d)\},\quad \langle x\rangle = \left(1+|x|^2\right)^{1/2},$$

we have, for fixed $\varepsilon>0$, $\psi^\varepsilon_{|t=0}\in\Sigma$, and the Cauchy problem is globally well-posed, $\psi^\varepsilon\in C(\mathbb{R}_t;\Sigma)$ (see e.g., [9]). It was established in [11] that the value

$$\alpha_c = 1+\frac{d\sigma}{2}$$

is critical in terms of the effect of the nonlinearity in the semi-classical limit $\varepsilon\to 0$. If $\alpha>\alpha_c$, then $\varphi^\varepsilon_{\rm lin}$, given by (1.9)-(1.10), is still a good approximation of $\psi^\varepsilon$ at least up to time of order $c\ln\frac{1}{\varepsilon}$. On the other hand, if $\alpha=\alpha_c$, nonlinear effects alter the behavior of $\psi^\varepsilon$ at leading order, through its envelope only. Replacing (1.9) by

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\left\langle Q(t)y,y\right\rangle u + |u|^{2\sigma}u, \tag{1.11}$$

and keeping the relation (1.10), $\varphi^\varepsilon$ is now a good approximation of $\psi^\varepsilon$. In [11] though, the time of validity of the approximation is not always proven to be of order at least $c\ln\frac{1}{\varepsilon}$,