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A TWO-PHASE FREE BOUNDARY PROBLEM FOR HARMONIC MEASURE

BY MAX ENGELSTEIN

ABSTRACT. – We study a 2-phase free boundary problem for harmonic measure first considered by Kenig and Toro [21] and prove a sharp Hölder regularity result. The central difficulty is that there is no a priori non-degeneracy in the free boundary condition. Thus we must establish non-degeneracy by means of monotonicity formulae.

RÉSUMÉ. – On étudie un problème dans la frontière libre avec deux phases, initialement examiné par Kenig et Toro [21], et on montre un résultat précis de régularité de Hölder. La difficulté essentielle est qu'il n'y a pas de conditions a priori de non-dégénérescence dans la condition de frontière libre. Par conséquent, nous devons déduire la non-dégénérescence en utilisant des formules de monotonie.

1. Introduction

In this paper we consider the following two-phase free boundary problem for harmonic measure: let Ω^+ be an unbounded 2-sided non-tangentially accessible (NTA) domain (see Definition 2.1) such that $\log(h)$ is regular, e.g., $\log(h) \in C^{0,\alpha}(\partial\Omega)$. Here $h := \frac{d\omega^-}{d\omega^+}$ and $\omega^\pm$ is the harmonic measure associated to the domain $\Omega^\pm$ ($\Omega^- := \mathrm{int}((\Omega^+)^c)$). We ask the question: what can be said about the regularity of $\partial\Omega$?

This question was first considered by Kenig and Toro (see [21]) when $\log(h) \in \mathrm{VMO}(d\omega^+)$. They concluded, under the initial assumption of δ-Reifenberg flatness, that Ω is a vanishing Reifenberg flat domain (see Definition 2.2). Later, the same problem, without the initial flatness assumption, was investigated by Kenig, Preiss and Toro (see [22]) and Badger (see [6] and [7]). Our work is a natural extension of theirs, though the techniques involved are substantially different.

Our main theorem is:

THEOREM 1.1. – *Let Ω be a 2-sided NTA domain with $\log(h) \in C^{k,\alpha}(\partial\Omega)$ where $k \geq 0$ is an integer and $\alpha \in (0,1)$.*

– *When $n = 2$: $\partial\Omega$ is locally given by the graph of a $C^{k+1,\alpha}$ function.*

- *When* $n \geq 3$*: there is some* $\delta_n > 0$ *such that if* $\delta < \delta_n$ *and* Ω *is* δ*-Reifenberg flat, then* $\partial\Omega$ *is locally given by the graph of a* $C^{k+1,\alpha}$ *function.*

Similarly, if $\log(h) \in C^\infty$ *or* $\log(h)$ *is analytic we can conclude (under the same flatness assumptions above) that* $\partial\Omega$ *is locally given by the graph of a* C^∞ *(resp. analytic) function.*

When $n > 2$, the initial flatness assumption is needed; if $n \geq 4$,

$$\Omega = \{X \in \mathbb{R}^n \mid x_1^2 + x_2^2 > x_3^2 + x_4^2\}$$

is a 2-sided NTA domain such that $\omega^+ = \omega^-$ on $\partial\Omega$ (where the poles are at infinity). As such, $h \equiv 1$ but, at zero, this domain is not a graph. In $\mathbb{R}^3$, H. Lewy (see [28]) proved that, for k odd, there are homogeneous harmonic polynomials of degree k whose zero set divides $\mathbb{S}^2$ into two domains. The cones over these regions are NTA domains and one can calculate that $\log(h) = 0$. Again, at zero, $\partial\Omega$ cannot be written as a graph. However, these two examples suggest an alternative to the a priori flatness assumption.

Theorem 1.2. – *Let* Ω *be a Lipschitz domain (that is,* $\partial\Omega$ *can be locally written as the graph of a Lipschitz function) and let* h *satisfy the conditions of Theorem 1.1. Then the same conclusions hold.*

The corresponding one-phase problem, "Does regularity of the Poisson kernel imply regularity of the free boundary?", has been studied extensively. Alt and Caffarelli (see [4]) first showed, under suitable flatness assumptions, that $\log(\frac{d\omega}{d\sigma}) \in C^{0,\alpha}(\partial\Omega)$ implies $\partial\Omega$ is locally the graph of a $C^{1,s}$ function. Jerison (see [17]) showed $s = \alpha$ above and, furthermore, if $\log(\frac{d\omega}{d\sigma}) \in C^{1,\alpha}(\partial\Omega)$, then $\partial\Omega$ is locally the graph of a $C^{2,\alpha}$ function (from here, higher regularity follows from classical work of Kinderlehrer and Nirenberg, [25]). Later, Kenig and Toro (see [24]) considered when $\log(\frac{d\omega}{d\sigma}) \in \mathrm{VMO}(d\sigma)$ and concluded that $\partial\Omega$ is a vanishing chord-arc domain (see Definition 1.8 in [24]).

Two-phase elliptic problems are also an object of great interest. The paper of Alt, Caffarelli and Friedman (see [5]) studied an "additive" version of our problem. Later, Caffarelli (see [10] for part one of three) studied viscosity solutions to an elliptic free boundary problem similar to our own. This work was then extended to the non-homogenous setting by De Silva, Ferrari and Salsa (see [12]). It is important to note that, while our problem is related to those studied above, we cannot immediately apply any of their results. In each of the aforementioned works there is an a priori assumption of non-degeneracy built into the problem (either in the class of solutions considered or in the free boundary condition itself). Our problem has no such a priori assumption. Unsurprisingly, the bulk of our efforts goes into establishing non-degeneracy.

Even in the case of $n = 2$, where the powerful tools of complex analysis can be brought to bear, our non-degeneracy results seem to be new. We briefly summarize some previous work in this area: let Ω^+ be a simply connected domain bounded by a Jordan curve and $\Omega^- = \overline{\Omega^+}^c$. Then $\partial\Omega = G^+ \cup S^+ \cup N^+$ where

- $\omega^+(N^+) = 0$.
- $\omega^+ \ll \mathscr{H}^1 \ll \omega^+$ on G^+.
- Every point of G^+ is the vertex of a cone in Ω^+. Furthermore, if C^+ is the set of all cone points for Ω^+, then $\mathscr{H}^1(C^+\backslash G^+) = 0 = \omega^+(C^+\backslash G^+)$.

- $\mathscr{H}^1(S^+) = 0$.
- For ω^+ a.e $Q \in S^+$ we have $\limsup_{r\downarrow 0} \frac{\omega^+(B(Q,r))}{r} = +\infty$ and $\liminf_{r\downarrow 0} \frac{\omega^+(B(Q,r))}{r} = 0$

with a similar decomposition for ω^-. These results are due to works by Makarov, McMillan, Pommerenke and Choi. See Garnett and Marshall [13], Chapter 6 for an introductory treatment and more precise references.

In our context, that is where $\omega^+ \ll \omega^- \ll \omega^+$, Ω is a 2-sided NTA domain and $\log(h) \in C^{0,\alpha}(\partial\Omega)$, one can use the Beurling monotonicity formula (see Lemma 1 in [8]) to show $\limsup_{r\downarrow\infty} \frac{\omega^\pm(B(x,r))}{r} < \infty$. Therefore, $\omega^\pm(S^+ \cup S^-) = 0$ and we can write $\partial\Omega = \Gamma \cup N$ where $\omega^\pm(N) = 0$ and Γ is 1-rectifiable (i.e., the image of countably many Lipschitz maps) and has σ-finite $\mathscr{H}^1$-measure. This decomposition is implied for $n > 2$ by the results of Section 5. In order to prove increased regularity one must bound from below $\liminf_{r\downarrow 0} \frac{\omega^+(B(Q,r))}{r}$, which we do in Corollary 6.4 and seems to be an original contribution to the literature.

The approach is as follows: after establishing some initial facts about blowups and the Lipschitz continuity of the Green's function (Sections 3 and 4) we tackle the issue of degeneracy. Our main tools here are the monotonicity formulae of Almgren, Weiss and Monneau which we introduce in Section 5. Unfortunately, in our circumstances these functionals are not actually monotonic. However, and this is the key point, we show that they are "almost monotonic" (see, e.g., Theorem 5.8). More precisely, we bound the first derivative from below by a summable function. From here we quickly conclude pointwise non-degeneracy. In Section 6, we use the quantitative estimates of the previous section to prove uniform non-degeneracy and establish the C^1 regularity of the free boundary.

At this point the regularity theory developed by De Silva et al. (see [12]) and Kinderlehrer et al. (see [25] and [26]) can be used to produce the desired conclusion. However, these results cannot be applied directly and some additional work is required to adapt them to our situation. These arguments, while standard, do not seem to appear explicitly in the literature. Therefore, we present them in detail here. Section 7 adapts the iterative argument of De Silva, Ferrari and Salsa [12] to get $C^{1,s}$ regularity for the free boundary. In Section 8 we first describe how to establish optimal $C^{1,\alpha}$ regularity and then $C^{2,\alpha}$ regularity (in analogy to the aforementioned work of Jerison [17]). This is done through an estimate in the spirit of Agmon et al. ([1] and [2]) which is proven in the appendix. Higher regularity then follows easily.

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2. Notation and Definitions

Throughout this article $\Omega \subset \mathbb{R}^n$ is an open set and our object of study. For simplicity, $\Omega^+ := \Omega$ and $\Omega^- := \overline{\Omega}^c$. To avoid technicalities we will assume that $\Omega^\pm$ are both unbounded and let $u^\pm$ be the Green's function of $\Omega^\pm$ with a pole at ∞ (our methods and theorems apply