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CONTINUITY AND DISCONTINUITY OF THE BOUNDARY LAYER TAIL

BY WILLIAM M. FELDMAN AND INWON C. KIM

ABSTRACT. – We investigate the continuity properties of the homogenized boundary data $\overline{g}$ for oscillating Dirichlet boundary data problems. The homogenized boundary condition arises as the boundary layer tail of a problem set in a half-space. The continuity properties of this boundary layer tail depending on the normal direction of the half space play an important role in the homogenization process in general bounded domains. We show that, for a generic non-rotation-invariant operator and boundary data, $\overline{g}$ is discontinuous at *every* rational direction. In particular this implies that the continuity condition of Choi and Kim [16] is essentially sharp. On the other hand, when the condition of [16] holds, we show a Hölder modulus of continuity for $\overline{g}$. When the operator is linear we show that $\overline{g}$ is Hölder-$\frac{1}{d}$ up to a logarithmic factor. The proofs are based on a new geometric observation on the limiting behavior of $\overline{g}$ at rational directions, reducing to a class of two dimensional problems for projections of the homogenized operator.

RÉSUMÉ. – Nous étudions les propriétés de continuité des données sur les bords homogénéisées $\overline{g}$ pour des problèmes de Dirichlet avec des données oscillantes. La condition au bord homogénéisée se pose comme la queue de la couche limite d'un problème posé dans un demi-espace. Les propriétés de cette queue de la couche limite en fonction de la direction normale du demi-espace jouent un rôle important dans le processus d'homogénéisation dans des domaines bornés généraux. Nous montrons que, pour un opérateur non-rotation invariant générique et les données au bord, $\overline{g}$ est discontinu à chaque direction rationnelle. En particulier, cela implique que la condition de continuité de Choi et Kim [16] est essentiellement sharp. D'autre part, lorsque la condition de [16] est satisfaite, nous montrons un module de continuité Hölder pour $\overline{g}$. Lorsque l'opérateur est linéaire, nous montrons que $\overline{g}$ est $\frac{1}{d}$−Hölder jusqu'à un facteur logarithmique. Les preuves sont basées sur une nouvelle observation géométrique sur le comportement limite de $\overline{g}$ dans des directions rationnelles, ce qui réduit à une classe de problèmes deux dimensionnelles pour les projections de l'opérateur homogénéisé.

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1. Introduction

To motivate the questions considered in this paper, let us start by discussing the homogenization of oscillating Dirichlet boundary data problems,

$$(1.1) \qquad \begin{cases} F(D^2u^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in} \quad U \\ u^\varepsilon = g(\frac{x}{\varepsilon}) & \text{on} \quad \partial U. \end{cases}$$

Here $F(M, y)$ is uniformly elliptic and positively 1-homogeneous in M, $g(y)$ is continuous, and both are $\mathbb{Z}^d$-periodic in y. In the linear case we are considering operators of the form $F(M, y) = -\mathrm{Tr}(A(y)M)$ with $1 \leq A(y) \leq \Lambda$ and $\mathbb{Z}^d$-periodic in y. There is no problem to include a large scale x dependence in g but we omit it here for clarity.

This type of problem has a singular behavior near boundary points x with inward normal direction ν_x aligned with a $\mathbb{Z}^d$-lattice vector, called *rational* directions. In order to mitigate the effects of the singularities the bounded domain $U \subset \mathbb{R}^d$ is typically assumed to be uniformly convex, although more general assumptions which rule out large flat portions of ∂U are also be sufficient for the results discussed below, see [19, 16]. In such domains it is known due to Feldman [19] that there exists $\overline{g} : S^{d-1} \to \mathbb{R}$, continuous at irrational directions, so that u^ε converges to $\overline{u}$ locally uniformly in U where $\overline{u}$ is the unique solution of,

$$(1.2) \qquad \begin{cases} \overline{F}(D^2\overline{u}) = 0 & \text{in} \quad U \\ \overline{u} = \overline{g}(\nu_x) & \text{on} \quad \partial U, \end{cases}$$

where, again, ν_x is the inward normal of U at $x \in \partial U$. Similar results have been obtained for linear divergence form equations starting with the work of Gérard-Varet and Masmoudi [21, 22] and continued by several authors [28, 2, 3, 4].

In this paper we study the continuity properties of the homogenized boundary data $\overline{g}$ by investigating the associated *cell problem* (1.4). Besides being a natural question on its own, continuity properties of $\overline{g}$ play an important role in obtaining rates of convergence for (1.2). In fact, if we could obtain Lipschitz continuity of $\overline{g}$ for the linear problem then we could also obtain an optimal rate of convergence that matches the rate for Laplacian operator. To our knowledge this particular connection has not been written down explicitly in the literature, however it is implicit in the methods used in several works [21, 22, 20]. Let us point out that the typical strategy to study homogenization of (1.1) is by ensuring that the impact coming from singular boundary points are negligible: this is because in the linear case zero measure sets are not seen by the Poisson kernel, and in the nonlinear case an analogous argument applies for boundary sets of small Hausdorff dimension. In contrast, here we investigate the behavior of $\overline{g}$ as ν_x approaches rational directions. In the linear case we show, interestingly, that $\overline{g}$ extends continuously to the rational directions, and in the nonlinear case we show that discontinuity is generic.

In the Neumann case the continuity of the corresponding $\overline{g}$ has been studied by Choi-Kim-Lee and Choi-Kim [15, 16]. There it was shown that when the averaged operator $\bar{F}$ is rotation invariant, homogenization holds and the homogenized boundary data is continuous. Following these works [19] showed homogenization for general F in the Dirichlet setting, due to the new observation that (1.2) has a unique solution if the discontinuity set of $\overline{g}(\nu_x)$ on ∂U has sufficiently small Hausdorff dimension. This brings up the natural question of whether the homogenized boundary condition could in fact be discontinuous when $\bar{F}$ is not rotation-invariant. Our main results are (i) an explicit estimate on the mode of continuity for $\overline{g}$ when $\overline{F}$ is rotation invariant or linear, (ii) when $\overline{F}$ is not rotation invariant or linear, $\overline{g}$ is 'generically' discontinuous at every boundary point with rational normal direction (see Theorem 1.3 and Corollary 1.4). These results seem to be new even in the linear case.

We expect our main results in this paper to hold with parallel proofs in both the Dirichlet and Neumann case. On the other hand we hope to keep our illustration simple so that our main ideas are presented clearly. For this reason we will only discuss the Dirichlet problem, even though our arguments build on the framework introduced for the Neumann problem in [16]. We leave the task of proving parallel results for the Neumann problem, including the general homogenization results in [19], for the future work.

We proceed to give a more precise, but still informal, derivation of (1.4) from (1.2). We begin by reminding the reader of the derivation of the cell problem determining $\overline{g}$. We consider a rescaling of the solution u^ε of (1.1) near a boundary point $x_0 \in \partial U$ with unit inner normal ν_{x_0},

$$v^\varepsilon(y) = u^\varepsilon(x_0 + \varepsilon y).$$

The limit of $v^\varepsilon(R\nu_{x_0})$ as $R \to \infty$ and $\varepsilon \to 0$, if it exists, will be the homogenized boundary data $\bar{g}(\nu_{x_0})$ as long as $\varepsilon R \to 0$. The behavior of v^ε outside of the oscillating boundary layer is the quantity of interest. To proceed with the analysis we inspect the equation solved by the v^ε,

$$\begin{cases} F(D^2 v^\varepsilon, y + \varepsilon^{-1}x_0) = 0 & \text{in } \ \varepsilon^{-1}(U - x_0) \\ v^\varepsilon = g(y + \varepsilon^{-1}x_0) & \text{on } \ \varepsilon^{-1}(\partial U - x_0). \end{cases} \tag{1.3}$$

Since F and g are assumed to be $\mathbb{Z}^d$ periodic in y, $\varepsilon^{-1}x_0$ can be replaced by $\tau_\varepsilon = \varepsilon^{-1}x_0$ mod $\mathbb{Z}^d$. Note that along various subsequences τ_ε could converge to any $\tau \in [0,1)^d$. This motivates the definition of the cell problem. Let $\nu \in S^{d-1}$, $\tau \in [0,1)^d$ and ψ be a continuous $\mathbb{Z}^d$-periodic function and define $v_{\nu,\tau}(\cdot;(\psi, F)) : P_\nu \to \mathbb{R}$ to solve,

$$\begin{cases} F(D^2 v_{\nu,\tau}, y + \tau) = 0 & \text{in } \ P_\nu := \{y \cdot \nu > 0\} \\ v_{\nu,\tau} = \psi(y + \tau) & \text{on } \ \partial P_\nu. \end{cases} \tag{1.4}$$

It is not too difficult to see, at least formally, that,

$$|v^\varepsilon(y) - v_{\nu_{x_0},\tau_\varepsilon}(y; g(x_0,\cdot))| \to 0 \ \text{ as } \ \varepsilon \to 0.$$

From this identification we can replace understanding $\bar{g}(\nu)$ with the easier problem of understanding the limit $v_{\nu,\tau}(R\nu)$ as $R \to \infty$ for every $\tau \in [0,1)^d$.

For irrational directions ν the distribution of g on $P_\nu + \tau\nu$ is, in an appropriate sense, invariant with respect to τ. For this reason it was possible to show, in [15, 16, 19], that, for irrational directions ν, there exists a limit $\mu(\nu, \psi, F)$, the so-called *boundary layer tail* of $v_{\nu,\tau}$, such that

$$\sup_{\tau \in [0,1)^d} \sup_{y \in \partial P_\nu} |v_{\nu,\tau}(y + R\nu; \psi) - \mu(\nu, \psi, F)| \to 0 \ \text{ as } \ R \to \infty. \tag{1.5}$$