ISSN 0012-9593

quatrième série - tome 50 fascicule 5 septembre-octobre 2017

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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C^1-*rigidity of circle maps with breaks for almost all rotation numbers*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

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Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE
de 1883 à 1888 par H. DEBRAY
de 1889 à 1900 par C. HERMITE
de 1901 à 1917 par G. DARBOUX
de 1918 à 1941 par É. PICARD
de 1942 à 1967 par P. MONTEL

Comité de rédaction au 1er janvier 2017

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Édition / *Publication*

Société Mathématique de France
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Fax : (33) 01 40 46 90 96

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email : `smf@smf.univ-mrs.fr`

Tarifs

Europe : 519 €. Hors Europe : 548 €. Vente au numéro : 77 €.

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ISSN 0012-9593

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 50, 2017, p. 1163 à 1203

C^1-RIGIDITY OF CIRCLE MAPS WITH BREAKS FOR ALMOST ALL ROTATION NUMBERS

BY KONSTANTIN KHANIN, SAŠA KOCIĆ AND ELIO MAZZEO

ABSTRACT. – We prove that, for almost all irrational $\rho \in (0,1)$, every two $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphisms with a break point, i.e., a singular point where the derivative has a jump discontinuity, with the same rotation number ρ and the same size of the break $c \in \mathbb{R}_+\backslash\{1\}$, are C^1-smoothly conjugate to each other.

RÉSUMÉ. – Nous démontrons que pour presque tous les irrationnels $\rho \in (0,1)$, deux difféomorphismes du cercle $C^{2+\alpha}$ lisses, $\alpha \in (0,1)$, avec un point de singularité de type rupture où la dérivée a une discontinuité de saut, avec le même nombre de rotation ρ et la même taille de rupture $c \in \mathbb{R}_+\backslash\{1\}$, sont C^1-conjugués l'un à l'autre.

1. Introduction

This paper establishes generic C^1-rigidity for circle diffeomorphisms with breaks. The result can viewed as a one-parameter extension of Herman's theory on the linearization of circle diffeomorphisms.

The problem of smoothness of a conjugacy to a linear rotation for smooth diffeomorphisms of a circle is a classic problem in dynamical systems. It was proven by Arnol′d [1], using the methods of KAM (Kolmogorov-Arnol′d-Moser) theory, that every analytic circle diffeomorphism with a Diophantine rotation number ρ, sufficiently close to the rigid rotation $R_\rho : x \mapsto x + \rho \mod 1$, is analytically conjugate to R_ρ. A number ρ is called Diophantine if there exists $C > 0$ and $\beta \geq 0$ such that $|\rho - p/q| > C/q^{2+\beta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Arnol′d also conjectured that the result remains true if the requirement of closeness to the rigid rotation is removed. A version of this global rigidity result, for smooth circle diffeomorphisms, was proven by Herman [6], and is the subject of classical Herman's theory. The theory was further developed for C^r-smooth maps, $r \geq 3$, by Yoccoz [20] who established a dependence of the regularity of the conjugacy on the Diophantine properties of the rotation numbers. For more recent work we refer the reader to [7, 11, 13, 18]. In a recent formulation [13], every $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphism, with rotation number

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ρ Diophantine with exponent $\beta < \alpha$, is $C^{1+\alpha-\beta}$-smoothly conjugate to the rotation R_ρ. Arnol′d also proved that this result cannot be extended to all irrational rotation numbers [1]. He constructed examples of analytic circle diffeomorphisms with irrational rotation numbers for which the invariant measure is singular, which implies that the conjugacy to the rigid rotation is not absolutely continuous.

We use the term *rigidity* for the phenomenon that any two maps within a given equivalence class determined by topological conjugacy are, in fact, C^1-smoothly conjugate to each other. Herman's theory establishes that, in the case of smooth circle diffeomorphisms, rigidity is guaranteed when rotation numbers satisfy a Diophantine condition. Over the last two decades, great effort has been made to understand the rigidity properties of circle diffeomorphisms with a singular point where the diffeomorphism condition is violated. The singular points refer either to points where the derivative vanishes (critical points) or where it has a jump discontinuity (break points). In the case of critical circle maps, i.e., circle maps with a single singular point where the derivative vanishes, the first rigidity results were obtained by de Faria and de Melo [4, 5]. They established the *convergence of renormalizations* — the main technical tool in proving rigidity results — and rigidity for analytic critical circle maps with the same irrational rotation number of bounded type (i.e., with bounded partial quotients) and the same (odd-integer) order of the critical point (i.e., the exponent of the power law behavior of the map in a neighborhood of the critical point). Renormalizations f_n of a circle map T are obtained from the restriction of T^{q_n} to a small interval, by an affine change of coordinates, where q_n is the denominator of the rational convergent p_n/q_n of the rotation number ρ (see next section). The convergence of renormalizations for analytic critical circle maps and for all irrational rotation numbers was later established by Yampolsky [19]. The results of de Faria and de Melo [4] show that even stronger $C^{1+\varepsilon}$-rigidity of analytic critical circle maps, for some $\varepsilon > 0$, is generic, i.e., it holds for almost all irrational rotation numbers. C^1-rigidity of analytic critical circle maps holds for all irrational rotation numbers, as was shown by Khanin and Teplinsky [12]. This phenomenon, when rigidity holds without any Diophantine-type conditions, is referred to as *robust rigidity*. Rigidity theory of non-analytic critical circle maps, however, has remained an open problem since, up to now, there is no proof of the convergence of renormalizations in this case.

The above results for critical circle maps suggested [8] that the rigidity might also be robust in the case of circle diffeomorphisms with a break point. In [8], rigidity was established for a set of rotation numbers of zero Lebesgue measure. However, as was shown by two of us [9], the above conjecture is false — robust rigidity does not hold for circle maps with breaks. We proved in [9] that there are irrational rotation numbers ρ, and pairs of analytic circle diffeomorphisms with breaks, with the same rotation number ρ and the same size of the break (i.e., the square root of the ratio of the left and right derivatives at the break point), for which any conjugacy between them is not even Lipschitz continuous. The question whether rigidity holds for typical rotation numbers, however, remained open. The main result of this paper provides an affirmative answer to this question.

Before we state our main result, let us define precisely the class of maps that we consider. A C^r-smooth circle diffeomorphism (map) with a break is a map $T : \mathbb{T}^1 \to \mathbb{T}^1$, $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, for which there exists $x_{\mathrm{br}} \in \mathbb{T}^1$ such that $T \in C^r([x_{\mathrm{br}}, x_{\mathrm{br}}+1])$; $T'(x)$ is bounded from below by a positive constant on $[x_{\mathrm{br}}, x_{\mathrm{br}}+1]$; the one-sided derivatives of T at x_{br} are such that the size of the break

$$c := \sqrt{\frac{T'_-(x_{\mathrm{br}})}{T'_+(x_{\mathrm{br}})}} \neq 1. \tag{1.1}$$

The main result of this paper is based on the following theorem.

THEOREM 1.1 ([10]). – *Let $\alpha \in (0,1)$ and let $c \in \mathbb{R}^+\backslash\{1\}$. There exists $\lambda \in (0,1)$ such that, for every two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break T and $\widetilde{T}$, with the same irrational rotation number $\rho \in (0,1)$, and the same size of the break c, there exists $C > 0$, such that the renormalizations f_n and $\widetilde{f_n}$ of T and $\widetilde{T}$, respectively, satisfy $\|f_n - \widetilde{f_n}\|_{C^2} \leq C\lambda^n$, for all $n \in \mathbb{N}$.*

REMARK 1. – This theorem establishes the exponential convergence of renormalizations for circle diffeomorphisms with a break, with a uniform rate λ for all irrational rotation numbers. Moreover, there exists $\mu \in (0,1)$, independent of α, such that $\lambda = \mu^\alpha$. This result is stronger than what is needed for our next theorem. Note that the statement of the theorem remains true if $c = 1$. This essentially follows from Herman's theory.

Let $\lambda_1 \in (\lambda, 1)$ and $C_1 > 0$. Let $S_e(C_1, \lambda_1)$ and $S_o(C_1, \lambda_1)$ be the sets of all irrational rotation numbers $\rho = [k_1, k_2, \dots] \in (0,1)$ whose subsequence of partial quotients k_{n+1} (see next section) for all n even or odd, respectively, satisfies the bound $k_{n+1} \leq C_1\lambda_1^{-n}$. Let $S_e(\lambda_1) := \bigcup_{C_1>0} S_e(C_1, \lambda_1)$ and $S_o(\lambda_1) := \bigcup_{C_1>0} S_o(C_1, \lambda_1)$. We define $S := S_e(\lambda_1)$, if $0 < c < 1$, and $S := S_o(\lambda_1)$, if $c > 1$. Theorem 1.1 and Theorem 2.2, proven in this paper, imply the following strong rigidity statement for circle diffeomorphisms with a break.

THEOREM 1.2. – *Any two $C^{2+\alpha}$-smooth, $\alpha \in (0,1)$, circle diffeomorphisms with a break T and $\widetilde{T}$, with the same size of the break $c \in \mathbb{R}_+\backslash\{1\}$ and the same rotation number $\rho \in S$, are C^1-smoothly conjugate to each other, i.e., there exists a C^1-smooth diffeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that $\varphi \circ T \circ \varphi^{-1} = \widetilde{T}$.*

REMARK 2. – Set S has full Lebesgue measure. One can also see that it contains some strongly Liouville numbers. The difference between the cases of odd and even n is related to a difference in the behavior of the renormalizations f_n, which will be explained in detail in the next section. If $0 < c < 1$ and n is even and sufficiently large or if $c > 1$ and n is odd and sufficiently large, the renormalizations f_n are concave and the renormalization parameter $a^{(n)} = f_n(0)$ (see the next section) can be exponentially small in k_{n+1}. If $0 < c < 1$ and n is odd and sufficiently large or if $c > 1$ and n is even and sufficiently large, the renormalizations f_n are convex and $a^{(n)}$ is bounded away from zero. The imposed condition on the rotation numbers controls the smallness of this parameter. It is not difficult to see that even the set of rotation numbers $\bigcap_{\lambda_1\in(0,1)} S_o(\lambda_1) \cap S_e(\lambda_1)$, for which rigidity holds for any $\alpha \in (0,1)$ and any $c \in \mathbb{R}^+\backslash\{1\}$, has full Lebesgue measure. On the other hand, it is not obvious that the sets $\bigcup_{\lambda_1\in(\lambda,1)} S_o(\lambda_1)$ and $\bigcup_{\lambda_1\in(\lambda,1)} S_e(\lambda_1)$, for which rigidity holds for some $\alpha \in (0,1)$ and $c \in \mathbb{R}_+\backslash\{1\}$, can be extended.