

*quatrième série - tome 50    fascicule 3    mai-juin 2017*

*ANNALES  
SCIENTIFIQUES  
de  
L'ÉCOLE  
NORMALE  
SUPÉRIEURE*

Tien-Cuong DINH & Xiaonan MA & Viêt-Anh NGUYÊN

*Equidistribution speed for Fekete points  
associated with an ample line bundle*

---

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

---

Publiées avec le concours du Centre National de la Recherche Scientifique

**Responsable du comité de rédaction / *Editor-in-chief***

Emmanuel KOWALSKI

**Publication fondée en 1864 par Louis Pasteur**

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE  
de 1883 à 1888 par H. DEBRAY  
de 1889 à 1900 par C. HERMITE  
de 1901 à 1917 par G. DARBOUX  
de 1918 à 1941 par É. PICARD  
de 1942 à 1967 par P. MONTEL

**Comité de rédaction au 1<sup>er</sup> janvier 2017**

P. BERNARD      A. NEVES  
S. BOUCKSOM    J. SZEFTEL  
E. BREUILLARD   S. VŨ NGỌC  
R. CERF          A. WIENHARD  
G. CHENEVIER    G. WILLIAMSON  
E. KOWALSKI

**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,  
45, rue d'Ulm, 75230 Paris Cedex 05, France.  
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.  
[annales@ens.fr](mailto:annales@ens.fr)

---

**Édition / *Publication***

Société Mathématique de France  
Institut Henri Poincaré  
11, rue Pierre et Marie Curie  
75231 Paris Cedex 05  
Tél. : (33) 01 44 27 67 99  
Fax : (33) 01 40 46 90 96

**Abonnements / *Subscriptions***

Maison de la SMF  
Case 916 - Luminy  
13288 Marseille Cedex 09  
Fax : (33) 04 91 41 17 51  
email : [smf@smf.univ-mrs.fr](mailto:smf@smf.univ-mrs.fr)

**Tarifs**

Europe : 519 €. Hors Europe : 548 €. Vente au numéro : 77 €.

---

© 2017 Société Mathématique de France, Paris

En application de la loi du 1<sup>er</sup> juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).

*All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.*

---

ISSN 0012-9593

Directeur de la publication : Stéphane Seuret  
Périodicité : 6 n<sup>os</sup> / an

## EQUIDISTRIBUTION SPEED FOR FEKETE POINTS ASSOCIATED WITH AN AMPLE LINE BUNDLE

BY TIEN-CUONG DINH, XIAONAN MA AND VIÊT-ANH NGUYỄN

---

**ABSTRACT.** – Let  $K$  be the closure of a bounded open set with smooth boundary in  $\mathbb{C}^n$ . A Fekete configuration of order  $p$  for  $K$  is a finite subset of  $K$  maximizing the Vandermonde determinant associated with polynomials of degree  $\leq p$ . A recent theorem by Berman, Boucksom and Witt Nyström implies that Fekete configurations for  $K$  are asymptotically equidistributed with respect to a canonical equilibrium measure, as  $p \rightarrow \infty$ . We give here an explicit estimate for the speed of convergence. The result also holds in a general setting of Fekete points associated with an ample line bundle over a projective manifold. Our approach requires a new estimate on Bergman kernels for line bundles and quantitative results in pluripotential theory which are of independent interest.

**RÉSUMÉ.** – Soit  $K$  l'adhérence d'un ouvert borné à bord lisse dans  $\mathbb{C}^n$ . Une configuration de Fekete d'ordre  $p$  pour  $K$  est un sous-ensemble fini de  $K$  qui maximise le déterminant de Vandermonde associé aux polynômes de degré  $\leq p$ . Un théorème récent de Berman, Boucksom et Witt Nyström implique que les configurations de Fekete sont asymptotiquement équiréparties par rapport à une mesure d'équilibre canonique quand  $p \rightarrow \infty$ . Nous donnons ici une estimation précise de la vitesse de convergence. Le résultat est aussi valable dans un cadre général des points de Fekete associés à un fibré en droites ample au-dessus d'une variété projective. Notre approche nécessite une estimation nouvelle sur les noyaux de Bergman pour les fibrés en droites et des résultats quantitatifs de la théorie du pluripotentiel qui sont d'intérêt indépendant.

*Notation.* – Throughout the paper,  $L$  denotes an ample holomorphic line bundle over a projective manifold  $X$  of dimension  $n$ . Fix also a smooth Hermitian metric  $h_0$  on  $L$  whose first Chern form, denoted by  $\omega_0$ , is a Kähler form. For simplicity, we use the Kähler metric on  $X$  induced by  $\omega_0$ . The induced distance is denoted by  $\text{dist}$ . Define  $\mu^0 := \|\omega_0^n\|^{-1} \omega_0^n$  the probability measure associated with the volume form  $\omega_0^n$ . The space of holomorphic sections of  $L^p := L^{\otimes p}$ , the  $p$ -th power of  $L$ , is denoted by  $H^0(X, L^p)$ . Its dimension is denoted by  $N_p$ . The metric  $h_0$  induces, in a canonical way, metrics on the line bundle  $L^p$  over  $X$ , the

---

T.-C. D. partially supported by Start-Up Grant R-146-000-204-133 from National University of Singapore. X. M. partially supported by Institut Universitaire de France, ANR-14-CE25-0012-01 and funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative. V.-A. N. partially supported by the Max-Planck institute for mathematics in Bonn.

vector bundle of the product  $L^p \times \dots \times L^p$  ( $N_p$  times) over  $X^{N_p}$ , and the determinant of the last one which is a line bundle over  $X^{N_p}$  and denoted by  $(L^p)^{\boxtimes N_p}$ . For simplicity, the norm, induced by  $h_0$ , of a section of these vector bundles is denoted by  $|\cdot|$ .

A general singular metric on  $L$  has the form  $h = e^{-2\psi} h_0$ , where  $\psi$  is an integrable function on  $X$  with values in  $\mathbb{R} \cup \{\pm\infty\}$ . Such a function  $\psi$  is called a *weight*. It also induces singular metrics on the above vector bundles, and we denote by  $|\cdot|_{p\psi}$  the corresponding norm of a section of  $L^p$  or the associated determinant line bundle over  $X^{N_p}$ . This is a function on  $X$  or  $X^{N_p}$  respectively. If  $K$  is a subset of  $X$ , the supremum on  $K$  or  $K^{N_p}$  of this function is denoted by  $\|\cdot\|_{L^\infty(K, p\psi)}$  or  $\|\cdot\|_{L^\infty(K^{N_p}, p\psi)}$ . Its  $L^2(\mu)$  or  $L^2(\mu^{\boxtimes N_p})$ -norm is denoted by  $\|\cdot\|_{L^2(\mu, p\psi)}$  or  $\|\cdot\|_{L^2(\mu^{\boxtimes N_p}, p\psi)}$ , where  $\mu$  is a probability measure on  $X$ . We sometimes drop the power  $N_p$  for simplicity. In the same way, we often add the index “ $\psi$ ” or “ $p\psi$ ,” if necessary, to inform the use of the weight  $\psi$  for  $L$  and hence  $p\psi$  for  $L^p$ .

The notations  $\rho_p(\mu, \phi)$ ,  $\mathcal{B}_p(\mu, \phi)$  will be introduced in Subsection 2.3,  $\mathcal{B}_p^\infty(K, \phi)$ ,  $\mathcal{B}_p^2(\mu, \phi)$ ,  $\mathcal{L}_p(K, \phi)$ ,  $\mathcal{L}_p(\mu, \phi)$ ,  $\mathcal{E}(\phi)$ ,  $\mathcal{E}_{\text{eq}}(K, \phi)$  in Subsection 3.1, and  $\mathcal{V}_p(\phi_1, \phi_2)$ ,  $\mathcal{W}(\phi_1, \phi_2)$ ,  $\varepsilon_p$ ,  $\mathcal{D}_p(K, \phi)$  in Subsection 3.2. Let  $\mathbb{B}(x, r)$  denote the ball of center  $x$  and radius  $r$  in  $X$  or in an Euclidean space. Similarly,  $\mathbb{D}(x, r)$  is the disk of center  $x$  and radius  $r$  in  $\mathbb{C}$ ,  $\mathbb{D}_r := \mathbb{D}(0, r)$  and  $\mathbb{D} := \mathbb{D}(0, 1)$ . The Lebesgue measure on an Euclidean space is denoted by  $\text{Leb}$ . The operators  $d^c$  and  $dd^c$  are defined by

$$d^c := \frac{\sqrt{-1}}{2\pi}(\bar{\partial} - \partial) \quad \text{and} \quad dd^c := \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}.$$

For  $m \in \mathbb{N}$  and  $0 < \alpha \leq 1$ ,  $\mathcal{C}^{m, \alpha}$  is the class of  $\mathcal{C}^m$  functions/differential forms whose partial derivatives of order  $m$  are Hölder continuous with Hölder exponent  $\alpha$ . We have  $\mathcal{C}^{m, \alpha} = \mathcal{C}^{m+\alpha}$  except for  $\alpha = 1$ . We use the natural norms on these spaces and for simplicity, define  $\|\cdot\|_m := 1 + \|\cdot\|_{\mathcal{C}^m}$  and  $\|\cdot\|_{m, \alpha} := 1 + \|\cdot\|_{\mathcal{C}^{m, \alpha}}$ . Denote by  $\text{Lip}$  the space of Lipschitz functions which is also equal to  $\mathcal{C}^{0,1}$  and by  $\widetilde{\text{Lip}}$  the space of functions  $v$  such that  $|v(x) - v(y)| \lesssim -\text{dist}(x, y) \log \text{dist}(x, y)$  for  $x, y$  close enough. We endow the last space with the norm

$$\|v\|_{\widetilde{\text{Lip}}} := \|v\|_\infty + \inf \{A \geq 0 : |v(x) - v(y)| \leq -A \text{dist}(x, y) \log \text{dist}(x, y) \text{ if } \text{dist}(x, y) \leq 1/2\}.$$

A function  $\phi : X \rightarrow \mathbb{R} \cup \{-\infty\}$  is called *quasi-plurisubharmonic* (quasi-p.s.h. for short) if it is locally the sum of a plurisubharmonic (p.s.h. for short) and a smooth function. A quasi-p.s.h. function  $\phi$  is called  $\omega_0$ -p.s.h. if  $dd^c\phi + \omega_0 \geq 0$  in the sense of currents. Denote by  $\text{PSH}(X, \omega_0)$  the set of such functions. If  $\phi$  is a bounded function in  $\text{PSH}(X, \omega_0)$ , define the associated Monge-Ampère measure and normalized Monge-Ampère measure by

$$\text{MA}(\phi) := (dd^c\phi + \omega_0)^n \quad \text{and} \quad \text{NMA}(\phi) := \|\text{MA}(\phi)\|^{-1} \text{MA}(\phi).$$

So  $\text{MA}(\phi)$  is a positive measure and  $\text{NMA}(\phi)$  is a probability measure on  $X$ . A quasi-p.s.h. function  $\phi$  is called *strictly  $\omega_0$ -p.s.h.* if  $dd^c\phi + \omega_0$  is larger than a Kähler form in the sense of currents, see [11, 14] for the basic notions and results of pluripotential theory.

*Some remarks.* – The constants involved in our computations below may depend on  $X, L, h_0$  and hence on  $\omega_0$  and  $\mu^0$ . However, they do not depend on the other weights used for the line bundle  $L$  but only on the upper bounds of suitable norms ( $\mathcal{C}^\alpha, \widetilde{\text{Lip}}, \dots$ ) of these weights. This property can be directly seen in our arguments. For simplicity, we will not repeat it in

each step of the proofs. The notations  $\gtrsim$  and  $\lesssim$  mean inequalities up to a positive multiple constant.

## 1. Introduction

Let  $K$  be a non-pluripolar compact subset of  $\mathbb{C}^n$ . The pluricomplex Green function of  $K$ , denoted by  $V_K^*(z)$ , is the upper-semicontinuous regularization of the Siciak-Zahariuta extremal function

$$V_K(z) := \sup \{u(z) : u \text{ p.s.h. on } \mathbb{C}^n, u|_K \leq 0, u(w) - \log \|w\| = O(1) \text{ as } w \rightarrow \infty\}.$$

This function  $V_K^*$  is locally bounded, p.s.h. and  $(dd^c V_K^*)^n$  defines a probability measure with support in  $K$ . It is called the equilibrium measure of  $K$  and denoted by  $\mu_{\text{eq}}(K)$ , see [29, 32].

Let  $\mathcal{P}_p$  be the set of holomorphic polynomials of degree  $\leq p$  on  $\mathbb{C}^n$ . This is a complex vector space of dimension

$$N_p := \binom{p+n}{n} = \frac{1}{n!} p^n + O(p^{n-1}).$$

Let  $(e_1, \dots, e_{N_p})$  be a basis of  $\mathcal{P}_p$ . Define for  $P = (x_1, \dots, x_{N_p}) \in (\mathbb{C}^n)^{N_p}$  the Vandermonde determinant  $W(P)$  by

$$W(P) := \det \begin{pmatrix} e_1(x_1) & \dots & e_1(x_{N_p}) \\ \vdots & \ddots & \vdots \\ e_{N_p}(x_1) & \dots & e_{N_p}(x_{N_p}) \end{pmatrix}.$$

A point  $P \in K^{N_p}$  is called a Fekete configuration for  $K$  if the function  $|W(\cdot)|$ , restricted to  $K^{N_p}$ , achieves its maximal value at  $P$ . It is not difficult to check that this definition does not depend on the choice of the basis  $(e_1, \dots, e_{N_p})$ , see [28].

Recently, Berman, Boucksom and Witt Nyström have proved that Fekete points  $x_1, \dots, x_{N_p}$  are asymptotically equidistributed with respect to the equilibrium measure  $\mu_{\text{eq}}(K)$  as  $p$  tends to infinity [3]. This property had been conjectured for quite some time, probably going back to the pioneering work of Leja in [19, 20], where the dimension 1 case was obtained. See also [22, 28] for more recent references on this topic. More precisely, let

$$\mu_p := \frac{1}{N_p} \sum_{j=1}^{N_p} \delta_{x_j}$$

denote the probability measure equidistributed on  $x_1, \dots, x_{N_p}$ . We call it a Fekete measure of order  $p$ . The above equidistribution result says that in the weak-\* topology

$$\lim_{p \rightarrow \infty} \mu_p = \mu_{\text{eq}}(K).$$

In fact, this theorem by Berman, Boucksom and Witt Nyström holds in a more general context of Fekete points associated with a line bundle. We will discuss this case later together with an interesting new approach by Ameur, Lev and Ortega-Cerdà [1, 21].

Fekete points are well known to be useful in several problems in mathematics and mathematical physics. It is therefore important to study the speed of the above convergence. For