ISSN 0012-9593

quatrième série - tome 51 fascicule 6 novembre-décembre 2018

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

John LESIEUTRE

Some constraints on positive entropy automorphisms of smooth threefolds

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 540 €. Hors Europe : 595 € ($ 863). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 51, 2018, p. 1507 à 1547

SOME CONSTRAINTS ON POSITIVE ENTROPY AUTOMORPHISMS OF SMOOTH THREEFOLDS

BY JOHN LESIEUTRE

ABSTRACT. – Suppose that X is a smooth, projective threefold over $\mathbb{C}$ and that $\phi : X \to X$ is an automorphism of positive entropy. We show that one of the following must hold, after replacing ϕ by an iterate: i) the canonical class of X is numerically trivial; ii) ϕ is imprimitive; iii) ϕ is not dynamically minimal. As a consequence, we show that if a smooth threefold M does not admit a primitive automorphism of positive entropy, then no variety constructed by a sequence of smooth blow-ups of M can admit a primitive automorphism of positive entropy.

In explaining why the method does not apply to threefolds with terminal singularities, we exhibit a non-uniruled, terminal threefold X with infinitely many K_X-negative extremal rays on $\overline{\mathrm{NE}}(X)$.

RÉSUMÉ. – Soit X une variété projective lisse de dimension trois sur $\mathbb{C}$. Nous supposons qu'il existe un automorphisme $\phi : X \dashrightarrow X$ d'entropie positive. Quitte à remplacer ϕ par un de ses itérés ϕ^n, nous montrons qu'une des affirmations suivantes sera verifiée : i) la classe canonique de X est numériquement triviale ; ii) ϕ est imprimitive ; iii) ϕ n'est pas dynamiquement minimal. Comme corollaire, nous montrons que si une variété lisse M de dimension trois n'admet pas d'automorphisme primitif d'entropie positive, il en est de même pour toute variété construite par une suite d'éclatements lisses de M.

Notre méthode ne s'applique pas dans le cadre des variétés à singularités terminales. Ceci sera illustré par l'exemple d'une variété uniréglée X qui admet une infinité de rayons extrémaux K_X-négatifs sur $\overline{\mathrm{NE}}(X)$.

1. Introduction

Suppose that X is a smooth projective variety over $\mathbb{C}$. An automorphism $\phi : X \to X$ is said to have *positive entropy* if the pullback map $\phi^* : N^1(X) \to N^1(X)$ has an eigenvalue greater than 1. By a fundamental result of Gromov and Yomdin, this notion of positive entropy coincides with the one familiar in dynamical systems, related to the separation of orbits by ϕ; we refer to [23] for an excellent survey of these results.

Although there are many interesting examples of positive entropy automorphisms of projective surfaces, examples in higher dimensions remain scarce. Our aim in this note is

 doi:10.24033/asens.2380

to give some constraints on the geometry of smooth, projective threefolds that admit automorphisms of positive entropy and partly explain this scarcity. These constraints are specific to automorphisms of threefolds: they hold neither for automorphisms of surfaces, nor for pseudoautomorphisms of threefolds.

Before stating the main results, we recall two basic ways in which an automorphism of X can be built out of automorphisms of "simpler" varieties.

DEFINITION 1.1. – An automorphism $\phi : X \to X$ is *imprimitive* if there exists a variety V with $1 \leq \dim V < \dim X$, a birational automorphism $\psi : V \dashrightarrow V$, and a dominant rational map $\pi : X \dashrightarrow V$ such that $\pi \circ \phi = \psi \circ \pi$. The map ϕ is called *primitive* if it is not imprimitive [33].

For example, if $\psi : V \to V$ is a positive entropy automorphism, the induced map $\phi : \mathbb{P}(TV) \to \mathbb{P}(TV)$ of the projectivized tangent bundle also has positive entropy, but is not primitive.

DEFINITION 1.2. – An automorphism $\phi : X \to X$ is *not dynamically minimal* if there exists a variety Y with terminal singularities, a birational morphism $\pi : X \to Y$, and an automorphism $\psi : Y \to Y$ with $\pi \circ \phi = \psi \circ \pi$. If no such $\pi : X \to Y$ exists, ϕ is called *dynamically minimal*.

For example, if $\psi : Y \to Y$ is a positive entropy automorphism, and $V \subset Y$ is a ψ-invariant subvariety, there is an induced automorphism $\phi : \mathrm{Bl}_V Y \to \mathrm{Bl}_V Y$. The map ϕ has positive entropy, but it is not dynamically minimal.

The restriction that Y have terminal singularities is quite natural from the point of view of birational geometry, for these are the singularities that can arise in running the minimal model program (MMP) on X. In dimension 2, having terminal singularities is equivalent to smoothness, and dynamical minimality is equivalent to the non-existence of ϕ-periodic (-1)-curves on X.

Positive entropy automorphisms of projective surfaces are in many respects well-understood. If X is a smooth projective surface admitting a positive entropy automorphism, it must be a blow-up of either $\mathbb{P}^2$, a K3 surface, an abelian surface, or an Enriques surface [8, Prop. 1]. Blow-ups of $\mathbb{P}^2$ at 10 or more points have proved to be an especially fertile source of examples, beginning with work of Bedford and Kim [3] and McMullen [21]. However, in higher dimensions, there are very few examples known of primitive, positive entropy automorphisms. The first such example on a smooth, rational threefold was given only recently by Oguiso and Truong [24, Theorem 1.4].

One result of this note is that the three-dimensional analogs of the basic blow-up constructions in dimension two can never yield primitive, positive entropy automorphisms.

THEOREM 1.3. – *Suppose that M is a smooth projective threefold that does not admit any automorphism of positive entropy, and that X is constructed by a sequence of blow-ups of M along smooth centers. Then any positive entropy automorphism $\phi : X \to X$ is imprimitive.*

This provides a partial answer to a question of Bedford:

QUESTION 1 (Bedford, cf. [29]). – *Does there exist a smooth blow-up of* $\mathbb{P}^3$ *admitting a positive entropy automorphism?*

According to Theorem 1.3, if such an automorphism exists, it must be imprimitive. Truong has also obtained many results on this question, showing that if X is constructed by a sequence of blow-ups of points and curves whose normal bundles satisfy certain constraints, then X admits no positive entropy automorphisms, and that under certain weaker conditions, any positive entropy automorphism has equal first and second dynamical degrees [29].

Whereas every smooth projective surface can be obtained as the blow-up of a minimal surface, this is far from true for threefolds. Although the sharpest results we obtain are in this blow-up setting, in combination with classification results from the MMP, the same approach yields some constraints on positive entropy automorphisms of arbitrary smooth threefolds for which the canonical class is not numerically trivial.

THEOREM 1.4. – *Suppose that X is a smooth projective threefold and that $\phi : X \to X$ is an automorphism of positive entropy. After replacing ϕ by some iterate, at least one of the following must hold:*

(1) the canonical class of X is numerically trivial;
(2) ϕ is imprimitive;
(3) ϕ is not dynamically minimal.

The conclusions in all these cases can be refined considerably; a more detailed subdivision into seven cases appears as Theorem 1.7 below. In Section 2, we show that all seven cases do occur, and the list can not be shortened.

We caution that Theorem 1.4 should not be construed as a classification of threefolds admitting a primitive automorphism of positive entropy. The chief difficulty lies in case (3): when ϕ is not dynamically minimal, the new variety Y on which ϕ induces an automorphism may no longer be smooth, so the result can not be applied inductively. This leads to the following.

COROLLARY 1.5. – *Suppose that $\phi : X \to X$ is a primitive, positive entropy automorphism of a smooth, projective, rationally connected threefold. Then there exists a non-smooth threefold Y with terminal singularities and a birational map $\pi : X \to Y$ such that some iterate of ϕ descends to an automorphism of Y.*

Experience with the MMP suggests that it is unsurprising that even in studying automorphisms of smooth threefolds, it is useful to consider threefolds with terminal singularities. The unexpected feature of Corollary 1.5 is that such singularities on Y are not only allowed, but unavoidable.

Results of Zhang show that if X admits a primitive, positive entropy automorphism, it must be either rationally connected or birational to a variety with numerically trivial canonical class [33, Theorem 1.2]. Our results are primarily of interest in the rationally connected setting, and are in some sense complementary to those of Zhang: although we obtain no new information on the birational type of X, we give some constraints on the geometry of a birational model on which ϕ acts as an automorphism. For example, we obtain