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*Scattering resonances for highly oscillatory potentials*

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# SCATTERING RESONANCES FOR HIGHLY OSCILLATORY POTENTIALS

BY ALEXIS DROUOT

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**ABSTRACT.** – We study resonances of compactly supported potentials  $V_\varepsilon(x) = W(x, x/\varepsilon)$  where  $W : \mathbb{R}^d \times \mathbb{R}^d / (2\pi\mathbb{Z})^d \rightarrow \mathbb{C}$ ,  $d$  odd. That means that  $V_\varepsilon$  is a sum of a slowly varying potential,  $W_0$ , and one oscillating at frequency  $1/\varepsilon$ . When  $W_0 \equiv 0$  we prove that there are no resonances above the line  $\text{Im } \lambda = -A \ln(\varepsilon^{-1})$ , except a simple resonance near 0 when  $d = 1$ . We show that this result is optimal by constructing a one-dimensional example. This settles a conjecture of Duchêne-Vukićević-Weinstein [12]. When  $W_0 \neq 0$  and  $W$  smooth we prove that resonances in fixed strips admit an expansion in powers of  $\varepsilon$ . The argument provides a method for computing the coefficients of the expansion. We produce an effective potential converging uniformly to  $W_0$  as  $\varepsilon \rightarrow 0$  and whose resonances approach resonances of  $V_\varepsilon$  modulo  $O(\varepsilon^4)$ . This improves the one-dimensional result of Duchêne, Vukićević and Weinstein and extends it to all odd dimensions.

**RÉSUMÉ.** – Nous étudions les résonances de potentiels à support compact  $V_\varepsilon(x) = W(x, x/\varepsilon)$ , où  $W : \mathbb{R}^d \times \mathbb{R}^d / (2\pi\mathbb{Z})^d \rightarrow \mathbb{C}$  et  $d$  est impair. Ainsi,  $V_\varepsilon$  est la somme d'un potentiel qui varie lentement  $W_0$  et d'un potentiel qui oscille à fréquence  $1/\varepsilon$ . Quand  $W_0 \equiv 0$  nous prouvons que  $V_\varepsilon$  n'a pas de résonances dans la zone  $\{\text{Im } \lambda \geq -A \ln(\varepsilon^{-1})\}$  mise à part une unique résonance proche de 0 si  $d = 1$ . Nous montrons par un exemple explicite que ce résultat est optimal. Cela prouve une conjecture de Duchêne-Vukićević-Weinstein [12]. Quand  $W_0 \neq 0$  et  $W$  est lisse nous montrons que les résonances de  $V_\varepsilon$  qui restent bornées lorsque  $\varepsilon$  tend vers 0 admettent une expansion en puissances de  $\varepsilon$ . Les arguments de la preuve permettent de calculer les coefficients de cette expansion. Nous construisons un potentiel effectif qui converge uniformément vers  $W_0$  lorsque  $\varepsilon$  tend vers 0 et dont les résonances sont à distance  $O(\varepsilon^4)$  de celles de  $W_0$ . Cela améliore et étend les résultats de Duchêne, Vukićević et Weinstein à toutes les dimensions impaires.

## 1. Introduction

In this paper we are interested in the poles of the meromorphic continuation of  $(-\Delta + \mathcal{V} - \lambda^2)^{-1}$  where  $d$  is odd and  $\mathcal{V} : \mathbb{R}^d \rightarrow \mathbb{C}$  is a bounded compactly supported potential. These poles called scattering resonances appear in many physical situations, for instance their imaginary parts are the rates of decay of waves scattered by  $\mathcal{V}$ .

Let  $-\Delta \geq 0$  be the free Laplacian on  $\mathbb{R}^d$ . The operator  $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ , well defined as an operator  $L^2(\mathbb{R}^d) \rightarrow H^2(\mathbb{R}^d)$  for  $\text{Im } \lambda > 0$ , extends to a meromorphic family of bounded operators  $L^2_{\text{comp}}(\mathbb{R}^d) \rightarrow H^2_{\text{loc}}(\mathbb{R}^d)$  for  $\lambda \in \mathbb{C}$  (see §1.5 for review of notation). This family admits one simple pole at 0 if  $d = 1$  and is entire if  $d \geq 3$ . If  $\mathcal{V}$  is a bounded compactly supported function on  $\mathbb{R}^d$  then  $R_{\mathcal{V}}(\lambda) = (-\Delta + \mathcal{V} - \lambda^2)^{-1}$  is well defined for  $\text{Im } \lambda \gg 1$  as an operator  $L^2(\mathbb{R}^d) \rightarrow H^2(\mathbb{R}^d)$ . It extends to a meromorphic family of operators  $L^2_{\text{comp}}(\mathbb{R}^d) \rightarrow H^2_{\text{loc}}(\mathbb{R}^d)$ . In this sense, the resonances of a real-valued potential  $\mathcal{V}$ —similarly, the poles of the meromorphic continuation of  $R_{\mathcal{V}}(\lambda)$ —are a generalization of eigenvalues of  $-\Delta + \mathcal{V}$ : each eigenvalue  $E$  of  $-\Delta + \mathcal{V}$  is negative and generates a resonance  $i\sqrt{-E}$ , and conversely every resonance  $\lambda$  of  $\mathcal{V}$  in the upper half-plane lies in  $i[0, \infty)$  and corresponds to the eigenvalue  $\lambda^2$ . Resonances of  $\mathcal{V}$  in the lower half-plane are not related to eigenvalues of  $-\Delta + \mathcal{V}$ , though they quantize the rate of decay of waves scattered by  $\mathcal{V}$ . We refer to [15, §2, 3] for a complete introduction to resonances in potential scattering.

Let  $W$  be a *bounded* complex valued function with support in  $\mathbb{B}^d(0, L) \times \mathbb{T}^d$ . We define  $V_\varepsilon$  as

$$V_\varepsilon(x) = W\left(x, \frac{x}{\varepsilon}\right).$$

If  $W$  is formally given by

$$W(x, y) = \sum_{k \in \mathbb{Z}^d} W_k(x) e^{iky}$$

we can write  $V_\varepsilon$  as a highly oscillatory perturbation of  $W_0$ :

$$(1.1) \quad V_\varepsilon(x) = W_0(x) + V_\#(x), \quad V_\#(x) = \sum_{k \neq 0} W_k(x) e^{ikx/\varepsilon}.$$

In this paper we study resonances of potentials  $V_\varepsilon$  given by (1.1).

### 1.1. Main results

The first theorem concerns the case of a vanishing slowly varying part. In the notations of (1.1) we will assume for this result that  $W \in L^\infty_0(\mathbb{B}^d(0, L) \times \mathbb{T}^d)$  (i.e.,  $\text{supp}(W)$  is a compact subset of  $\mathbb{B}^d(0, L) \times \mathbb{T}^d$  and  $W$  is uniformly bounded) and that moreover,

$$(1.2) \quad \begin{aligned} \exists s \in (0, 1), \quad \sum_{k \neq 0} \frac{|W_k|_{H^s}}{|k|^s} &< \infty \text{ if } d = 1, \\ \sum_{k \neq 0} \frac{\|W_k\|_1}{|k|} &< \infty \text{ if } d \geq 3. \end{aligned}$$

**THEOREM 1.** – *Let  $W$  be in  $L^\infty_0(\mathbb{B}^d(0, L) \times \mathbb{T}^d, \mathbb{C})$  such that  $W_0 \equiv 0$  and (1.2) holds. Then there exists  $C, c, A$  three positive constants such that*

$$\begin{aligned} \text{if } d = 1, \text{ Res}(V_\varepsilon) \setminus \mathbb{D}(0, c\varepsilon^{s/2}) &\subset \{\lambda \in \mathbb{C} : \text{Im } \lambda \leq C - A \ln(\varepsilon^{-1})\}; \\ \text{if } d \geq 3, \text{ Res}(V_\varepsilon) &\subset \{\lambda \in \mathbb{C} : \text{Im } \lambda \leq C - A \ln(\varepsilon^{-1})\}. \end{aligned}$$

This settles a conjecture of [12]: for odd dimensions  $d \geq 3$  and  $\varepsilon$  small enough the potential  $V_\varepsilon$  does not have a bound state. In §2.3 we construct a step-like function  $W$  such that  $V_{\pi/(2n)}$  has a resonance  $\lambda_n \sim -i \ln(n)$  as  $n \rightarrow +\infty$ . This shows that one cannot improve the rate of escape of resonances given by Theorem 1 in dimension 1.

In the next statements we always assume that  $W$  is smooth. We consider the case  $W_0 \neq 0$ . If  $\lambda_0$  is a simple resonance of  $W_0$  we can write

$$(1.3) \quad R_{W_0}(\lambda) = \frac{i u \otimes v}{\lambda - \lambda_0} + H(\lambda), \quad H(\lambda) \text{ holomorphic near } \lambda_0,$$

for some functions  $u, v \in H^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C})$  called resonant states. As the potential  $V_\varepsilon$  given by (1.1) converges weakly to  $W_0$ , it is natural to expect that resonances of  $V_\varepsilon$  converge to resonances of  $W_0$ . In fact a much stronger statement holds:

**THEOREM 2.** – *Let  $W$  belong to  $C^\infty(\mathbb{B}^d(0, L) \times \mathbb{T}^d, \mathbb{C})$  and  $V_\varepsilon$  be given by (1.1). Let  $\lambda_0$  be a simple resonance of  $W_0$ . In a neighborhood of  $\lambda_0$  and for  $\varepsilon$  small enough the potential  $V_\varepsilon$  admits a unique resonance  $\lambda_\varepsilon$ . Moreover, for any  $N$ ,*

$$\lambda_\varepsilon = \lambda_0 + c_2 \varepsilon^2 + c_3 \varepsilon^3 + \dots + c_{N-1} \varepsilon^{N-1} + O(\varepsilon^N), \quad c_j \in \mathbb{C}.$$

If  $u, v$  are the resonant states of (1.3) then

$$(1.4) \quad \begin{aligned} c_2 &= i \int_{\mathbb{R}^d} \Lambda_0(x) u(x) v(x) dx, & c_3 &= i \int_{\mathbb{R}^d} \Lambda_1(x) u(x) v(x) dx, \\ \Lambda_0 &= \sum_{k \neq 0} \frac{W_k W_{-k}}{|k|^2}, & \Lambda_1 &= -2 \sum_{k \neq 0} \frac{W_{-k} ((k \cdot D) W_k)}{|k|^4}. \end{aligned}$$

If  $W$  is real-valued then so are  $\Lambda_0$  and  $\Lambda_1$ . In §3.1 we will prove a version of Theorem 2 for resonances of higher multiplicity. Theorem 2 implies that perturbations of  $W_0$  by a high frequency potential  $V_\#$  enjoy some similarities with suitable analytic perturbations of  $W_0$ . In fact we have the following

**THEOREM 3.** – *Assume that  $W$  belongs to  $C^\infty(\mathbb{B}^d(0, L) \times \mathbb{T}^d, \mathbb{C})$  and that  $V_\varepsilon$  is given by (1.1). Let  $V_{\text{eff},\varepsilon} = W_0 - \varepsilon^2 \Lambda_0 - \varepsilon^3 \Lambda_1$  where  $\Lambda_0, \Lambda_1$  are given in (1.4). For every bounded family  $\varepsilon \mapsto \mu_\varepsilon$  of simple resonances of  $V_{\text{eff},\varepsilon}$  there exists a family of resonances  $\varepsilon \mapsto \lambda_\varepsilon$  of  $V_\varepsilon$  such that*

$$|\lambda_\varepsilon - \mu_\varepsilon| = O(\varepsilon^4).$$

*Conversely for every bounded family  $\varepsilon \mapsto \lambda_\varepsilon$  of simple resonances of  $V_\varepsilon$  there exists a family of resonances  $\varepsilon \mapsto \mu_\varepsilon$  of  $V_{\text{eff},\varepsilon}$  such that*

$$|\lambda_\varepsilon - \mu_\varepsilon| = O(\varepsilon^4).$$

The potential  $V_{\text{eff},\varepsilon}$  plays the role of an effective potential. In dimension one  $\Lambda_0$  was already derived in [12].

We next give a uniform description of the behavior of resonances of  $V_\varepsilon$  as  $\varepsilon \rightarrow 0$ . For  $W_0 \in C^\infty(\mathbb{B}^d(0, L), \mathbb{C})$  we define  $m_{W_0}(\lambda_0)$  the multiplicity of a resonance  $\lambda_0$  of  $W_0$ . If