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Tomoyuki ABE & Hélène ESNAULT

*A Lefschetz theorem for overconvergent isocrystals with Frobenius structure*

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Annales Scientifiques de l'École Normale Supérieure,

45, rue d'Ulm, 75230 Paris Cedex 05, France.

Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.

[annales@ens.fr](mailto:annales@ens.fr)

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Société Mathématique de France

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# A LEFSCHETZ THEOREM FOR OVERCONVERGENT ISOCRYSTALS WITH FROBENIUS STRUCTURE

BY TOMOYUKI ABE AND HÉLÈNE ESNAULT

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**ABSTRACT.** – We show a Lefschetz theorem for irreducible overconvergent  $F$ -isocrystals on smooth varieties defined over a finite field. We derive several consequences from it.

**RÉSUMÉ.** – Nous montrons un théorème de Lefschetz pour les  $F$ -isocristaux surconvergeants sur des variétés lisses définies sur un corps fini. Nous en tirons plusieurs conséquences.

## Introduction

Let  $X_0$  be a normal geometrically connected scheme of finite type defined over a finite field  $\mathbb{F}_q$ , let  $\mathcal{F}_0$  be an irreducible lisse Weil  $\overline{\mathbb{Q}}_\ell$ -sheaf with finite determinant (thus in fact  $\mathcal{F}_0$  is an étale sheaf as well), where  $\ell \neq p = \text{char}(\mathbb{F}_q)$ . In Weil II [8, Conj. 1.2.10], Deligne conjectured the following.

- (i) The sheaf  $\mathcal{F}_0$  is of weight 0.
- (ii) There is a number field  $E \subset \overline{\mathbb{Q}}_\ell$  such that for any  $n > 0$  and  $x \in X_0(\mathbb{F}_{q^n})$ , the characteristic polynomial  $f_x(\mathcal{F}_0, t) := \det(1 - tF_x \mid \mathcal{F}_{0,\bar{x}})$  lies in  $E[t]$ , where  $F_x$  is the geometric Frobenius of  $x$ .
- (iii) For any  $\ell' \neq p$  and any embedding  $\sigma: E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$ , for any  $n > 0$  and  $x \in X_0(\mathbb{F}_{q^n})$ , any root of  $\sigma f_x(\mathcal{F}_0, t)$  is an  $\ell'$ -adic unit.
- (iv) For any  $\sigma$  as in (iii), there is an irreducible  $\overline{\mathbb{Q}}_{\ell'}$ -lisse sheaf  $\mathcal{F}_{0,\sigma}$ , called the  $\sigma$ -companion, such that  $\sigma f_x(\mathcal{F}_0, t) = f_x(\mathcal{F}_{0,\sigma}, t)$ .
- (v) There is a crystalline version of (iv).

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Deligne's conjectures (i)–(iv) have been proved by Lafforgue [18, Thm. VII.6] when  $X_0$  is a smooth curve, as a corollary of the Langlands correspondence, which is proven showing that automorphic forms are in some sense motivic.

When  $X_0$  has dimension at least 2, the automorphic side on which one could rely to prove Deligne's conjectures is not available: there is no theory of automorphic forms in higher dimension. The problem then becomes how to reduce, by geometry, the statements to dimension 1. For (i) and (iii), one proves a Lefschetz theorem (see [12, Thm. 2.15], [10, 1.5–1.9], [13, B1]):

0.1. THEOREM. – *On  $X_0$  smooth, for any closed point  $x_0$ , there exists a smooth curve  $C_0$  and a morphism  $C_0 \rightarrow X_0$  such that  $x_0 \rightarrow X_0$  lifts to  $x_0 \rightarrow C_0$ , and such that the restriction of  $\mathcal{F}_0$  to  $C_0$  remains irreducible.*

Using Theorem 0.1, Deligne proved (ii) ([10, Thm. 3.1]), and Drinfeld, using (ii), proved (iv) in ([12, Thm. 1.1]), assuming in addition  $X_0$  to be smooth. In particular Drinfeld proved in [12, Thm. 2.5] the following key theorem.

0.2. THEOREM. – *If  $X_0$  is smooth, given a number field  $E \subset \overline{\mathbb{Q}_\ell}$ , and a place  $\lambda$  of  $E$  dividing  $\ell$ , a collection of polynomials  $f_x(t) \in E[t]$  indexed by any  $n > 0$  and  $x \in X_0(\mathbb{F}_{q^n})$ , such that the following two conditions are satisfied:*

- (i) *for any smooth curve  $C_0$  with a morphism  $C_0 \rightarrow X_0$  and any  $n > 0$  and  $x \in C_0(\mathbb{F}_{q^n})$ , there exists a lisse étale  $\overline{\mathbb{Q}_\ell}$ -sheaf  $\mathcal{F}_0^{C_0}$  on  $C_0$  with monodromy in  $\mathrm{GL}(r, E_\lambda)$  such that  $f_x(t) = f_x(\mathcal{F}_0^{C_0}, t)$ , where  $E_\lambda$  is the completion of  $E$  with respect to the place  $\lambda$ ;*
- (ii) *there exists a finite étale cover  $X'_0 \rightarrow X_0$  such that  $\mathcal{F}_0^{C_0}$  is tame on all  $C_0$  factoring through  $X'_0 \rightarrow X_0$ .*

*Then there exists a lisse  $\overline{\mathbb{Q}_\ell}$ -sheaf  $\mathcal{F}_0$  on  $X_0$  with monodromy in  $\mathrm{GL}(r, E_\lambda)$ , such that for any  $n > 0$  and  $x \in X_0(\mathbb{F}_{q^n})$ ,  $f_x(t) = f_x(\mathcal{F}_0, t) \in E[t]$ .*

Further, to realize the assumptions of Theorem 0.2 in order to show the existence of  $\mathcal{F}_{0,\sigma}$ , Drinfeld uses Theorem 0.1 in [12, 4.1]. He constructs step by step the residual representations with monodromy in  $\mathrm{GL}(r, \mathcal{O}_{E_\lambda}/\mathfrak{m}^n)$  for  $n$  growing, where  $\mathcal{O}_{E_\lambda}$  is the ring of integers of  $E_\lambda$  and  $\mathfrak{m}$  is its maximal ideal.

The formulation of (v) has been made explicit by Crew [7, 4.13]. The conjecture is that the crystalline category analogous to the category of Weil  $\overline{\mathbb{Q}_\ell}$ -sheaves is the category of overconvergent  $F$ -isocrystals (see Section 1.1 for the definitions). In order to emphasize the analogy between  $\ell$  and  $p$ , one slightly reformulates the definition of companions. One replaces  $\sigma$  in (iii) by an isomorphism  $\sigma: \overline{\mathbb{Q}_\ell} \rightarrow \overline{\mathbb{Q}_{\ell'}}$  (see [13, Thm. 4.4]), and keeps (iv) as it is. Here  $\ell, \ell'$  are any two prime numbers. For  $\ell' = p, \ell \neq p$ , and  $\mathcal{F}$  an irreducible lisse  $\overline{\mathbb{Q}_\ell}$ -sheaf, one requests the existence of an overconvergent  $F$ -isocrystal  $M_0$  on  $X_0$  with eigenpolynomial  $f_x(M_0, t)$  such that  $f_x(M_0, t) = \sigma f_x(\mathcal{F}, t) \in \sigma(E)[t]$  for any  $n > 0$  and  $x \in X_0(\mathbb{F}_{q^n})$ , where  $f_x(\mathcal{F}, t)$  is the characteristic polynomial of the geometric Frobenius at  $x$  on  $\mathcal{F}$  (see Section 1.4 for the definitions). The isocrystal  $M_0$  is called a  $\sigma$ -companion to  $\mathcal{F}$ . Given an irreducible overconvergent  $F$ -isocrystal  $M_0$  with finite determinant on  $X_0$ , and  $\sigma$  as above, a lisse  $\ell$ -adic Weil sheaf  $\mathcal{F}$  on  $X_0$  is a  $\sigma^{-1}$ -companion if  $\sigma^{-1} f_x(M_0, t) = f_x(\mathcal{F}, t) \in E[t]$

at  $x \in X_0(\mathbb{F}_{q^n})$  (see Definition 1.4). Similarly we can assume  $p = \ell = \ell'$ . This way we can talk on  $\ell$ -adic or  $p$ -adic companions of either an  $M_0$  or an  $\mathcal{F}$ . The companion correspondence should preserve the notions of irreducibility, finiteness of the determinant, the eigenpolynomials at closed points of  $X_0$ , and the ramification.

The conjecture in the strong form has been proven by the first author when  $X_0$  is a smooth curve ([1, Intro. Thm.]). The aim of this article is to prove the following analog of Theorem 0.1 on  $X$  smooth.

0.3. THEOREM (Theorem 3.10). – *Let  $X_0$  be a smooth geometrically connected scheme over  $\mathbb{F}_q$ . Let  $M_0$  be an irreducible overconvergent  $F$ -isocrystal with finite determinant. Then for every closed point  $x_0 \rightarrow X_0$ , there exists a smooth irreducible curve  $C_0$  defined over  $k$ , together with a morphism  $C_0 \rightarrow X_0$  and a factorization  $x_0 \rightarrow C_0 \rightarrow X_0$ , such that the pull-back of  $M_0$  to  $C_0$  is irreducible.*

Theorem 0.3, together with [10, Rmk. 3.10], footnote 2, and [1, Thm. 4.2.2] enable one to conclude that there is a number field  $E \subset \overline{\mathbb{Q}_p}$  such that for any  $n > 0$  and  $x \in X_0(\mathbb{F}_{q^n})$ ,  $f_x(M_0, t) \in E[t]$  (see Lemma 4.1). This yields the  $p$ -adic analog of (i) over a smooth variety  $X_0$ . (See Section 4.6 when  $X_0$  is normal). Then Theorem 0.2 implies the existence of  $\ell$ -adic companions to a given irreducible overconvergent  $F$ -isocrystal  $M_0$  with finite determinant (see Theorem 4.2). We point out that the existence of  $\ell$ -adic companions has already been proven by Kedlaya in [17, Thm. 5.3] in a different way, using weights (see [17, §4, Intro.]), however not their irreducibility. The Lefschetz Theorem 3.10 implies that the companion correspondence preserves irreducibility.

Theorem 0.3 has other consequences (see Section 4), aside of the existence already mentioned of  $\ell$ -adic companions. Deligne's finiteness theorem [13, Thm. 1.1] transposes to the crystalline side (see Corollary 4.3): on  $X_0$  smooth, there are finitely many isomorphism classes of irreducible overconvergent  $F$ -isocrystals in bounded rank and bounded ramification, up to twist by a character of the finite field. One can also kill the ramification of an  $F$ -overconvergent isocrystal by a finite étale cover in Kedlaya's semistability reduction theorem (Remark 4.4).

We now explain the method of proof of Theorem 0.3. We replace  $M_0$  by the full Tannakian subcategory  $\langle M \rangle$  of the category of overconvergent  $F$ -isocrystals spanned by  $M$  over the algebraic closure  $\overline{\mathbb{F}_q}$  (we drop the lower indices  $_0$  to indicate this, see Section 1.1 for the definitions). We slightly improve the theorem ([11, Prop. 2.21 (a), Rmk. 2.29]) describing the surjectivity of an homomorphism of Tannaka groups in categorical terms in Lemma 1.6: the restriction functor  $\langle M \rangle \rightarrow \langle M|_C \rangle$  to a curve  $C \rightarrow X$  is an equivalence when it is fully faithful and any  $F$ -overconvergent isocrystal of rank 1 on  $C$  is torsion. Class field theory for  $F$ -overconvergent isocrystals ([2, Lem. 6.1]<sup>(1)</sup>) implies the torsion property. As for full faithfulness, the problem is of cohomological nature, one has to compute that the restriction homomorphism  $H^0(X, N) \rightarrow H^0(C, N|_C)$  is an isomorphism for all objects  $N$  in  $\langle M \rangle$ . In the tame case, this is performed in Section 2 using the techniques developed in [3]. As a corollary,  $\ell$ -adic companions exist in the tame case (see Proposition 2.8). In the wild case, Kedlaya's

<sup>(1)</sup> See Remark 4.6 for a correction of a mistake in this lemma.