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ACYLINDRICAL HYPERBOLICITY OF THE THREE-DIMENSIONAL TAME AUTOMORPHISM GROUP

BY STÉPHANE LAMY AND PIOTR PRZYTYCKI

ABSTRACT. – We prove that the group $\mathrm{STame}(\mathbf{k}^3)$ of special tame automorphisms of the affine 3-space is not simple, over any base field of characteristic zero. Our proof is based on the study of the geometry of a 2-dimensional simply-connected simplicial complex $\mathcal{C}$ on which the tame automorphism group acts naturally. We prove that $\mathcal{C}$ is contractible and Gromov-hyperbolic, and we prove that $\mathrm{Tame}(\mathbf{k}^3)$ is acylindrically hyperbolic by finding explicit loxodromic weakly proper discontinuous elements.

RÉSUMÉ. – Nous montrons que le groupe $\mathrm{STame}(\mathbf{k}^3)$ des automorphismes modérés unimodulaires de l'espace affine de dimension 3 n'est pas simple, sur tout corps de base de caractéristique zéro. Notre preuve repose sur l'étude géométrique d'un complexe simplicial $\mathcal{C}$ simplement connexe et de dimension 2, sur lequel le groupe des automorphismes modérés agit naturellement. Nous montrons que $\mathcal{C}$ est contractible et hyperbolique au sens de Gromov, puis nous prouvons que $\mathrm{Tame}(\mathbf{k}^3)$ est acylindriquement hyperbolique en exhibant des éléments loxodromiques satisfaisant la propriété WPD.

1. Introduction

The *tame automorphism group* of the affine space $\mathbf{k}^3$, over a base field $\mathbf{k}$ of characteristic zero, is the subgroup of the polynomial automorphism group $\mathrm{Aut}(\mathbf{k}^3)$ generated by the affine and elementary automorphisms:

$$\mathrm{Tame}(\mathbf{k}^3) = \langle A_3, E_3 \rangle,$$

where

$$\begin{aligned} A_3 &= \mathrm{GL}_3(\mathbf{k}) \ltimes \mathbf{k}^3, \text{ and} \\ E_3 &= \{(x_1, x_2, x_3) \mapsto (x_1 + P(x_2, x_3), x_2, x_3) \mid P \in \mathbf{k}[x_2, x_3]\}. \end{aligned}$$

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There is a natural homomorphism Jac: $\mathrm{Tame}(\mathbf{k}^3) \to \mathbf{k}^*$ given by the Jacobian determinant. The kernel $\mathrm{STame}(\mathbf{k}^3)$ of this homomorphism is the *special tame automorphism group*. Analogously one defines $\mathrm{Tame}(\mathbf{k}^n)$ and $\mathrm{STame}(\mathbf{k}^n)$ for arbitrary $n \geqslant 2$. It is a natural question whether $\mathrm{STame}(\mathbf{k}^n)$ is a simple group. In this paper we prove that $\mathrm{STame}(\mathbf{k}^3)$ is not simple (and indeed very far from being simple).

Our strategy is to use an action of $\mathrm{Tame}(\mathbf{k}^3)$ on a Gromov-hyperbolic triangle complex, and to exhibit a loxodromic weakly proper discontinuous element of $\mathrm{STame}(\mathbf{k}^3)$, in the sense of M. Bestvina and K. Fujiwara [1]. Recall that an isometry f of a metric space X is *loxodromic* if for some (hence any) $x \in X$ there exists $\lambda > 0$ such that for any $k \in \mathbb{Z}$ we have $|x, f^k \cdot x| \geqslant \lambda|k|$. Suppose that f belongs to a group G acting on X by isometries. We say that f is *weakly proper discontinuous* (WPD) if for some (hence any) $x \in X$ and any $C \geqslant 1$, for k sufficiently large there are only finitely many $j \in G$ satisfying $|x, j \cdot x| \leqslant C$ and $|f^k \cdot x, j \circ f^k \cdot x| \leqslant C$.

By the work of F. Dahmani, V. Guirardel, and D. Osin [8, Thm 8.7], the existence of an action of a non-virtually cyclic group G on a Gromov-hyperbolic metric space, with at least one loxodromic WPD element, implies that G has a free normal subgroup, and in particular G is not simple. By the work of D. Osin [18, Thm 1.2], such a group is *acylindrically hyperbolic*: there exists a (different) Gromov-hyperbolic space on which the action of G is acylindrical, a notion introduced for general metric spaces by B. Bowditch [3].

This strategy was recently applied to various transformation groups in algebraic geometry. We now review a few examples to explain how the group $\mathrm{Tame}(\mathbf{k}^3)$ fits in the global picture.

First we discuss the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$, the *Cremona group* of rank 2, which is the group of birational transformations of the projective plane. It is by no means obvious to find a Gromov-hyperbolic space on which the Cremona groups acts. One takes all projective surfaces dominating $\mathbb{P}^2_{\mathbf{k}}$ by a sequence of blow-ups, and considers the direct limit of their spaces of curves, called Néron-Severi groups. The limit is endowed with a lorentzian intersection form defining an infinite dimensional hyperboloid. This hyperboloid was introduced in [5] and used to prove for instance a Tits alternative for the Cremona group. Then it was used in [7] to prove the non-simplicity of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ over an algebraically closed field $\mathbf{k}$. Finally, the above mentioned strategy was applied in [14] to obtain the non-simplicity over an arbitrary base field.

Note that one of the initial motivations for [8] was the application to the mapping class group. As it is the case for the Cremona group, in studying the mapping class group one uses an action on a non-locally compact Gromov-hyperbolic space (the complex of curves), but the parallel goes beyond that. For instance, there are striking similarities between the notion of dilatation factor for a pseudo-Anosov map, and the dynamical degree of a generic Cremona map: see the survey [6] for more details.

The above results about $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ were inspired by previous work on its subgroup $\mathrm{Aut}(\mathbf{k}^2) = \mathrm{Tame}(\mathbf{k}^2)$. It is classical that $\mathrm{Aut}(\mathbf{k}^2)$ is the amalgamated product of two of its subgroups, and so we get an action of $\mathrm{Aut}(\mathbf{k}^2)$ on the associated Bass-Serre tree (which is obviously Gromov-hyperbolic). Together with some classical small cancelation theory this was used by V. Danilov [9] to produce many normal subgroups in $\mathrm{Aut}(\mathbb{C}^2)$ (and in $\mathrm{SAut}(\mathbb{C}^2)$), see also [10]. Recently these results were extended to the case of an arbitrary field by A. Minasyan and D. Osin [17, Cor 2.7], again by producing concrete examples of WPD elements.

When one tries to extend these results to higher dimensions, one has to face the formidable gap in complexity between birational geometry of surfaces and in higher dimension. We refer to the introduction of [2] for a few more comments on this side of the story. The group Tame($\mathbf{k}^3$) seems to be a good first step to enter the world of dimension 3. It was a classical question proposed by M. Nagata in the 70' whether the inclusion Tame($\mathbf{k}^3$) $\subset$ Aut($\mathbf{k}^3$) was strict. This was confirmed 30 years later by I.P. Shestakov and U.U. Umirbaev [20], with an argument which was somewhat simplified by S. Kuroda [12]. Then it was recently noticed [2, 21, 13] that we can rephrase the theory developped by these authors by saying that Tame($\mathbf{k}^3$) is the amalgamated product of three subgroups along their pairwise intersections. Equivalently, the group Tame($\mathbf{k}^3$) acts on a simply connected 2-dimensional simplicial complex $\mathcal{C}$, with fundamental domain a single triangle. This complex $\mathcal{C}$ is the main object of our present work (see Section 2 for the definition).

To end the historical background, note that the situation on the affine 3-dimensional quadric has been also successfully explored. (In fact, we considered it a test setting for the whole strategy, before trying to handle the affine space $\mathbf{k}^3$.) The notion of a tame automorphism in this context was introduced by S. Vénéreau and the first author. An action on a Gromov-hyperbolic CAT(0) square complex was constructed in [2]. WPD elements were recently produced by A. Martin [16].

Our first two results about the geometry of the complex $\mathcal{C}$ associated with Tame($\mathbf{k}^3$) are the following.

Theorem A. – *$\mathcal{C}$ is contractible.*

Theorem B. – *$\mathcal{C}$ is Gromov-hyperbolic.*

As is often the case when dealing with 2-dimensional complexes, our arguments rely on understanding disk diagrams, i.e., simplicial disks mapping to the complex. The main difficulty here is that, by contrast with the above mentioned settings, the complex $\mathcal{C}$ does not admit an equivariant CAT(0) metric. We circumvent this problem by a procedure of "transport of curvature". Precisely, given a disk diagram, first we assign to each triangle of the diagram angles $\pi/2, \pi/3, \pi/6$, and then, by putting an orientation on certain edges of the diagram, we describe how to transport any excess of positive curvature at a given vertex to neighboring vertices. In this sense we obtain that any disk diagram is negatively curved, which gives Theorem A, and also, via linear isoperimetric inequality, Theorem B.

We now turn to the existence of WPD elements in Tame($\mathbf{k}^3$). On the 1-skeleton $\mathcal{C}^1$ of $\mathcal{C}$ we use the path metric where each edge has length 1, which is quasi-isometric to any Tame($\mathbf{k}^3$)-equivariant path-metric on $\mathcal{C}$. Let $n \geqslant 0$, and let $g, h, f \in$ Tame($\mathbf{k}^3$) be the automorphisms defined by

$$\begin{aligned} g^{-1}(x_1, x_2, x_3) &= (x_2, x_1 + x_2 x_3, x_3), \\ h^{-1}(x_1, x_2, x_3) &= (x_3, x_1, x_2), \\ f &= g^n \circ h. \end{aligned}$$