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FIRST-PASSAGE PERCOLATION AND LOCAL MODIFICATIONS OF DISTANCES IN RANDOM TRIANGULATIONS

BY Nicolas CURIEN AND Jean-François LE GALL

ABSTRACT. – We study local modifications of the graph distance in large random triangulations. Our main results show that, in large scales, the modified distance behaves like a deterministic constant $\mathbf{c} \in (0, \infty)$ times the usual graph distance. This applies in particular to the first-passage percolation distance obtained by assigning independent random weights to the edges of the graph. We also consider the graph distance on the dual map, and the first-passage percolation on the dual map with exponential edge weights, which is closely related to the so-called Eden model. In the latter two cases, we are able to compute explicitly the constant $\mathbf{c}$ by using earlier results about asymptotics for the peeling process. In general however, the constant $\mathbf{c}$ is obtained from a subadditivity argument in the infinite half-plane model that describes the asymptotic shape of the triangulation near the boundary of a large ball. Our results apply in particular to the infinite random triangulation known as the UIPT, and show that balls of the UIPT for the modified distance are asymptotically close to balls for the graph distance.

RÉSUMÉ. – Nous étudions l'effet de perturbations locales de la distance de graphe dans les grandes triangulations planaires aléatoires. Nous montrons qu'à grande échelle, la nouvelle distance se comporte comme c fois la distance de graphe où c est une constante déterministe dépendant du type de la perturbation effectuée. Cela s'applique en particulier à la métrique de percolation de premier passage obtenue en donnant des longueurs i.i.d. à chaque arête, à la distance de graphe sur la carte duale et au modèle d'Eden (percolation de premier passage avec poids exponentiels sur la carte duale). Dans les deux derniers cas, nous pouvons même calculer explicitement la constante c en utilisant un lien avec le processus d'épluchage (peeling process). En général, la constante c reste inconnue et provient d'un argument de sous-additivité appliqué à un modèle infini de triangulation du demi-plan qui décrit la structure d'une grande triangulation aléatoire près du bord d'une grande boule centrée à l'origine. Nos résultats s'appliquent également à l'UIPT et montrent que les grandes boules pour la distance modifiée sont proches de boules pour la distance de graphe initiale.

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1. Introduction

In the recent years, there has been much effort to understand the large-scale geometry of random planar maps viewed as random metric spaces for the usual graph distance on their vertex set. A major achievement in the area is the construction and study of the so-called Brownian map, which has been proved to be the universal scaling limit of many different classes of planar maps equipped with the graph distance (see [27, 33] and more recently [1, 2, 12]). An account of these developments can be found in the surveys [28, 34]. In the present work, we replace the graph distance by other natural choices of distances on the vertex set or on the set of faces, and we show that, in large scales, these new distances behave like the original graph distance, up to a constant multiplicative factor. In particular, we prove that the vertex set of a uniformly distributed random plane triangulation with n vertices equipped with the (suitably rescaled) first-passage percolation distance obtained by assigning independent random lengths to the edges converges in distribution to the Brownian map as $n \to \infty$, in the Gromov–Hausdorff sense. So, in some sense, the extra randomness coming from the weights assigned to the edges plays no role in the limit. This is a new illustration of the universality of the Brownian map as a two-dimensional model of random geometry. We mention here that the study of discrete or continuous models of random geometry has been strongly motivated by their relevance to various domains of theoretical physics, and in particular to the so-called two-dimensional quantum gravity. Discrete random geometry has been the subject of intensive research in the physics literature since the beginning of the eighties when Polyakov suggested to solve questions coming from string theory and quantum gravity by developing a formalism to calculate sums over random surfaces, as a kind of analog of the famous Feynman path integrals. We refer to the book [4] for an overview of the use of discrete random surfaces in theoretical physics.

Let us turn to a more detailed description of our main results. We recall that a planar map is a proper embedding of a finite connected (multi)graph in the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms of the sphere. We will always consider rooted planar maps, which means that there is a distinguished oriented edge whose initial vertex is called the root vertex. The faces are the connected components of the complement of edges, and a planar map is called a triangulation if all its faces are triangles (possibly with two sides glued together).

1.0.0.1. *Modified distances*. – If m is a rooted planar map, we let $\mathsf{V}(\mathrm{m}), \mathsf{E}(\mathrm{m})$ and $\mathsf{F}(\mathrm{m})$ denote respectively the sets of vertices, edges and faces of m. The set $\mathsf{V}(\mathrm{m})$ is usually equipped with the graph distance, which is denoted by $\mathrm{d_{gr}}$ (or $\mathrm{d_{gr}^m}$ if there is a risk of ambiguity). We introduce the following modifications of the graph distance.

Case 0: First-passage (bond) percolation. Assign independent identically distributed positive random variables $w(e)$ to all edges $e \in \mathsf{E}(\mathrm{m})$. We assume that the common distribution of the "weights" $w(e)$ is supported on $[\kappa, 1]$ for some $\kappa \in (0, 1]$. The associated first-passage percolation distance is defined on $\mathsf{V}(\mathrm{m})$ by setting for any $x, y \in \mathsf{V}(\mathrm{m})$,

$$\mathrm{d_{fpp}}(x, y) = \inf_{\gamma : x \to y} \sum_{e \in \gamma} w(e),$$

where the infimum runs over all paths γ going from x to y in the map m.

Case 1: DUAL GRAPH DISTANCE. Consider the dual map $\mathrm{m}^\dagger$, whose vertices are the faces of m, and each edge e of m corresponds to an edge of $\mathrm{m}^\dagger$ connecting the two (possibly equal) faces incident to e. We may then consider the graph distance on $\mathsf{V}(\mathrm{m}^\dagger) = \mathsf{F}(\mathrm{m})$, which we denote by $\mathrm{d}^\dagger_{\mathrm{gr}}$.

Case 2: EDEN MODEL. This is the first-passage percolation model on $\mathrm{m}^\dagger$ corresponding to exponential edge weights. More precisely, we assign independent exponential random variables with parameter 1 to the edges of $\mathrm{m}^\dagger$ (or equivalently to the edges of m) and the associated first-passage percolation distance on $\mathsf{F}(\mathrm{m}) = \mathsf{V}(\mathrm{m}^\dagger)$ is denoted by $\mathrm{d}^\dagger_{\mathrm{Eden}}$. We call this the Eden model because of the close relation with the classical Eden growth model, see in particular [3, Section 6] or [36, Section 1.2.1], and also [18, Proposition 15].

The functions d_{gr} and $\mathrm{d}_{\mathrm{fpp}}$ are distances on $\mathsf{V}(\mathrm{m})$ whereas $\mathrm{d}^\dagger_{\mathrm{gr}}$ and $\mathrm{d}^\dagger_{\mathrm{Eden}}$ are distances on $\mathsf{F}(\mathrm{m})$. To compare the latter distances to the usual graph metric on m, we will replace faces by incident vertices, and we use the notation $x \lhd f$ to mean that the vertex x is incident to the face f.

1.0.0.2. *Finite triangulations*. – We consider these "modified distances" when $\mathrm{m} = \mathcal{T}_n$ is a random planar map chosen uniformly at random in the set of all rooted plane triangulations with $n+1$ vertices (we consider type I triangulations where loops and multiple edges are allowed). In each of the previous cases, we are able to prove that the modified distances behave in large scales like a deterministic constant times the graph distance on $\mathsf{V}(\mathcal{T}_n)$. More precisely, there exist constants $\mathbf{c}_0, \mathbf{c}_1$ and $\mathbf{c}_2$ in $(0,\infty)$ such that we have the following three convergences in probability

$$n^{-1/4} \sup_{x,y\in\mathsf{V}(\mathcal{T}_n)} \left|\mathrm{d}_{\mathrm{fpp}}(x,y) - \mathbf{c}_0 \cdot \mathrm{d}_{\mathrm{gr}}(x,y)\right| \xrightarrow[n\to\infty]{} 0, \tag{1}$$

$$n^{-1/4} \sup_{\substack{x,y\in\mathsf{V}(\mathcal{T}_n),\ f,g\in\mathsf{F}(\mathcal{T}_n)\\ x\lhd f \text{ and } y\lhd g}} \left|\mathrm{d}^\dagger_{\mathrm{gr}}(f,g) - \mathbf{c}_1 \cdot \mathrm{d}_{\mathrm{gr}}(x,y)\right| \xrightarrow[n\to\infty]{} 0, \tag{2}$$

$$n^{-1/4} \sup_{\substack{x,y\in\mathsf{V}(\mathcal{T}_n),\ f,g\in\mathsf{F}(\mathcal{T}_n)\\ x\lhd f \text{ and } y\lhd g}} \left|\mathrm{d}^\dagger_{\mathrm{Eden}}(f,g) - \mathbf{c}_2 \cdot \mathrm{d}_{\mathrm{gr}}(x,y)\right| \xrightarrow[n\to\infty]{} 0. \tag{3}$$

Since the convergence of rescaled triangulations to the Brownian map [27] implies that the typical graph distance between two vertices of $\mathcal{T}_n$ is of order $n^{1/4}$, the convergence (1) shows that in large scales $\mathrm{d}_{\mathrm{fpp}}(x,y)$ is proportional to $\mathrm{d}_{\mathrm{gr}}(x,y)$. In fact (1) implies that the set $\mathsf{V}(\mathcal{T}_n)$ equipped with the metric $n^{-1/4}\mathrm{d}_{\mathrm{fpp}}$ converges in distribution in the Gromov-Hausdorff sense to (a scaled version of) the Brownian map, and that this convergence takes place jointly with that of $(\mathsf{V}(\mathcal{T}_n), n^{-1/4}\mathrm{d}_{\mathrm{gr}})$ proved in [27] (see Corollary 23 below). Similarly (2) shows that uniform rooted trivalent maps with n faces (which are the dual maps of rooted triangulations with n vertices) converge after rescaling toward the Brownian map.

In case 0, the constant $\mathbf{c}_0$ depends on the distribution of the weights and an explicit calculation of this constant seems hopeless. However in cases 1 and 2 (dual graph and Eden model) the constants can be computed exactly and we have

$$\mathbf{c}_1 = 1 + 2\sqrt{3} \quad \text{and} \quad \mathbf{c}_2 = 2\sqrt{3}.$$