

*quatrième série - tome 53      fascicule 4      juillet-août 2020*

*ANNALES  
SCIENTIFIQUES  
de  
L'ÉCOLE  
NORMALE  
SUPÉRIEURE*

Daniel LOUGHRAN & Alexei N. SKOROBOGATOV & Arne  
SMEETS

*Pseudo-split fibers and arithmetic surjectivity*

---

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

---

Publiées avec le concours du Centre National de la Recherche Scientifique

**Responsable du comité de rédaction / *Editor-in-chief***

Patrick BERNARD

**Publication fondée en 1864 par Louis Pasteur**

**Comité de rédaction au 1<sup>er</sup> janvier 2020**

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE

P. BERNARD      D. HARARI

de 1883 à 1888 par H. DEBRAY

S. BOUCKSOM      A. NEVES

de 1889 à 1900 par C. HERMITE

G. CHENEVIER      J. SZEFTEL

de 1901 à 1917 par G. DARBOUX

Y. DE CORNULIER      S. VŪ NGỌC

de 1918 à 1941 par É. PICARD

A. DUCROS      A. WIENHARD

de 1942 à 1967 par P. MONTEL

G. GIACOMIN      G. WILLIAMSON

**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,

45, rue d'Ulm, 75230 Paris Cedex 05, France.

Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.

[annales@ens.fr](mailto:annales@ens.fr)

---

**Édition et abonnements / *Publication and subscriptions***

Société Mathématique de France

Case 916 - Luminy

13288 Marseille Cedex 09

Tél. : (33) 04 91 26 74 64

Fax : (33) 04 91 41 17 51

email : [abonnements@smf.emath.fr](mailto:abonnements@smf.emath.fr)

**Tarifs**

Abonnement électronique : 428 euros.

Abonnement avec supplément papier :

Europe : 576 €. Hors Europe : 657 € (\$ 985). Vente au numéro : 77 €.

---

© 2020 Société Mathématique de France, Paris

En application de la loi du 1<sup>er</sup> juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).

*All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.*

---

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand

Périodicité : 6 n<sup>os</sup> / an

# PSEUDO-SPLIT FIBERS AND ARITHMETIC SURJECTIVITY

BY DANIEL LOUGHRAN, ALEXEI N. SKOROBOGATOV  
AND ARNE SMEETS

---

**ABSTRACT.** – Let  $f : X \rightarrow Y$  be a dominant morphism of smooth, proper and geometrically integral varieties over a number field  $k$ , with geometrically integral generic fiber. We give a necessary and sufficient geometric criterion for the induced map  $X(k_v) \rightarrow Y(k_v)$  to be surjective for almost all places  $v$  of  $k$ . This generalizes a result of Denef which had previously been conjectured by Colliot-Thélène, and can be seen as an optimal geometric version of the celebrated Ax-Kochen theorem.

**RÉSUMÉ.** – Soit  $f : X \rightarrow Y$  un morphisme dominant de variétés lisses, propres et géométriquement intègres définies sur un corps de nombres  $k$ , dont la fibre générique est géométriquement intègre. Nous donnons un critère géométrique, à la fois nécessaire et suffisant, pour que l'application induite  $X(k_v) \rightarrow Y(k_v)$  soit surjective pour presque toute place  $v$  de  $k$ . Ceci généralise un résultat de Denef précédemment conjecturé par Colliot-Thélène. Notre résultat peut être vu comme une version géométrique optimale du célèbre théorème de Ax-Kochen.

## 1. Introduction

**1.1.** – A famous theorem of Ax-Kochen [6] states that any homogeneous polynomial over  $\mathbf{Q}_p$  of degree  $d$  in at least  $d^2 + 1$  variables has a non-trivial zero, provided that  $p$  avoids a certain finite exceptional set of primes depending only on  $d$ . This was originally proved using model theory. Denef recently found purely algebro-geometric proofs [12, 13]. In [13], he did so by proving a more general conjecture of Colliot-Thélène [8, §3, Conjecture].

The essential notion (first introduced by the second author in [32, Definition 0.1]) appearing in this conjecture is that of a *split scheme*:

**DEFINITION 1.1.** – Let  $k$  be a perfect field. A scheme  $X$  of finite type over  $k$  is called *split* if  $X$  contains an irreducible component of multiplicity 1 which is geometrically irreducible.

Here the *multiplicity* of an irreducible component  $Z$  of  $X$  is the length of the local ring of  $X$  at the generic point of  $Z$ . In particular, it has multiplicity 1 if and only if it is generically reduced. Denef's result [13, Theorem 1.2] is the following.

**THEOREM 1.2 (Denef).** – *Let  $f : X \rightarrow Y$  be a dominant morphism of smooth, proper, geometrically integral varieties over a number field  $k$ , with geometrically integral generic fiber. Assume that for every modification  $f' : X' \rightarrow Y'$  of  $f$  with  $X'$  and  $Y'$  smooth such that the generic fibers of  $f$  and  $f'$  are isomorphic, the fiber  $(f')^{-1}(D)$  is a split  $\kappa(D)$ -variety for every  $D \in (Y')^{(1)}$ .*

*Then  $Y(k_v) = f(X(k_v))$  for all but finitely many places  $v$  of  $k$ .*

Here  $k_v$  denotes the completion of  $k$  at the place  $v$ ,  $(Y')^{(1)}$  denotes the set of points of codimension 1 in  $Y'$ , and  $\kappa(D)$  is the residue field of  $D$ . A *modification* of  $f$  is a commutative diagram

$$(1.1) \quad \begin{array}{ccc} X' & \xrightarrow{\alpha_X} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\alpha_Y} & Y, \end{array}$$

where  $f' : X' \rightarrow Y'$  is a dominant morphism of proper and geometrically integral varieties over  $k$ , and  $\alpha_X : X' \rightarrow X$  and  $\alpha_Y : Y' \rightarrow Y$  are birational morphisms.

One obtains the Ax-Kochen theorem by applying Theorem 1.2 to the universal family of all hypersurfaces of degree  $d$  in  $\mathbf{P}^n$  with  $n \geq d^2$ ; that the hypotheses of the theorem are satisfied in this case was shown by Colliot-Thélène (see [8, Remarque 4]).

**1.2.** – In this paper we strengthen Denef's result, by determining conditions which are both *necessary and sufficient* for the map  $f : X(k_v) \rightarrow Y(k_v)$  to be surjective for almost all places  $v$ . Our result uses the following weakening of Definition 1.1 (in §2.2 we also give a more general definition over arbitrary ground fields).

**DEFINITION 1.3.** – Let  $k$  be a perfect field with algebraic closure  $\bar{k}$ . A scheme  $X$  of finite type over  $k$  is called *pseudo-split* if every element of  $\text{Gal}(\bar{k}/k)$  fixes some irreducible component of  $X \times_k \bar{k}$  of multiplicity 1.

It is clear that pseudo-splitness is weaker than splitness, the latter meaning that a *single* irreducible component of  $X \times_k \bar{k}$  of multiplicity 1 is fixed by *all* of  $\text{Gal}(\bar{k}/k)$ . With this terminology, we can state our generalization of Denef's result as follows:

**THEOREM 1.4.** – *Let  $k$  be a number field. Let  $f : X \rightarrow Y$  be a dominant morphism of smooth, proper, geometrically integral varieties over  $k$  with geometrically integral generic fiber. Then  $Y(k_v) = f(X(k_v))$  for all but finitely many places  $v$  of  $k$  if and only if for every modification  $f' : X' \rightarrow Y'$  of  $f$ , with  $X'$  and  $Y'$  smooth, and for every point  $D \in (Y')^{(1)}$ , the fiber  $(f')^{-1}(D)$  is a pseudo-split  $\kappa(D)$ -variety.*

In the notation introduced by the first and third named authors in their recent work [25, §3], the morphisms  $f : X \rightarrow Y$  satisfying the conclusion of the theorem are exactly the morphisms such that  $\Delta(f') = 0$  for every modification  $f'$  of  $f$ .

**1.3.** – Theorem 1.4 will be deduced from finer results. With  $f : X \rightarrow Y$  as in Theorem 1.2, Colliot-Thélène asked in [9, §13.1] how the geometry of  $f$  relates to the surjectivity of the map  $X(k_v) \rightarrow Y(k_v)$ , for a possibly infinite collection of places  $v$ . He called this phenomenon “surjectivité arithmétique” (note that this is different from the notion of arithmetic surjectivity studied in [16]). We develop general criteria which allow one to decide whether, for an *individual* (but large) place  $v$ , the map  $X(k_v) \rightarrow Y(k_v)$  is surjective. They involve certain invariants which we call “ $s$ -invariants,” defined in §3—local versions of the  $\delta$ -invariants introduced in [25, §3]; their definition is given in terms of the geometry of  $f$  and does not involve model theory.

The following result is proved in §6 using tools from logarithmic geometry, in particular, a logarithmic version of Hensel’s lemma and “weak toroidalisation”. It should be viewed as the main theorem of the paper and is a geometric criterion, in the style of Colliot-Thélène’s conjecture, for surjectivity of the map  $X(k_v) \rightarrow Y(k_v)$ .

**THEOREM 1.5.** – *Let  $k$  be a number field. Let  $f : X \rightarrow Y$  be a dominant morphism of smooth, proper, geometrically integral varieties over  $k$ , with geometrically integral generic fiber. Then there exist a modification  $f' : X' \rightarrow Y'$  of  $f$  with  $X'$  and  $Y'$  smooth, and a finite set of places  $S$  of  $k$  such that for all  $v \notin S$  the following are equivalent:*

- (1) *the map  $X(k_v) \rightarrow Y(k_v)$  is surjective;*
- (2) *for every codimension 1 point  $D' \in (Y')^{(1)}$ , we have  $s_{f', D'}(v) = 1$ .*

The invariants  $s_{f', D'}(v)$  appearing in the statement will be defined in §3. They are defined in terms of the Galois action on the irreducible components of the fiber of  $f'$  over  $D'$ . One benefit of our approach is that it yields a single model for  $f$  which can be used to test arithmetic surjectivity using a finite list of criteria.

A simple consequence of Theorem 1.5 is the following:

**THEOREM 1.6.** – *Let  $f : X \rightarrow Y$  be a dominant morphism of smooth, proper and geometrically integral varieties over a number field  $k$ , with geometrically integral generic fiber. The set of places  $v$  such that  $Y(k_v) = f(X(k_v))$  is Frobenian.*

Here we use the term “Frobenian” in the sense of Serre [31, §3.3] (see §3.1). Frobenian sets of places have a density, but being Frobenian is much stronger than just having a density; for example, an infinite Frobenian set has positive density. It is also possible to prove Theorem 1.6 using model-theoretic results and techniques such as quantifier elimination [5, 28]; our method avoids these and is completely algebro-geometric. However, we know of no model-theoretic proof of the finer Theorems 1.4 and 1.5. (From a model-theoretic perspective, one may view Theorem 1.5 as an explicit instance of quantifier elimination).

**1.4.** – Some of the ingredients of our proof are already present in the work of Denef [12, 13], e.g., the use of the weak toroidalisation theorem [4, 3]. We need more ingredients from logarithmic geometry, cf. §5—essentially a few basic properties of log smooth morphisms and log blow-ups. The choice of a log smooth model for the morphism makes some of its arithmetic properties more transparent, and can be seen as a convenient way to come up with a Galois stratification, in the sense of Fried and Sacerdote [14]. On the other hand, we also use work of Serre [31] on Frobenian functions, expanding upon what was done in [25].