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# ON THE LIFTING OF HILBERT CUSP FORMS TO HILBERT-SIEGEL CUSP FORMS

BY Tamotsu IKEDA AND Shunsuke YAMANA

ABSTRACT. – Starting from a Hilbert cusp form of weight $2\kappa$, we will construct a Hilbert-Siegel cusp form of weight $\kappa + \frac{m}{2}$ and degree $m$ and its transfer to inner forms of symplectic groups. Applications include a relation between Fourier coefficients of Hilbert cusp forms of weight $n + \frac{1}{2}$ and a certain weighted sum of the representation numbers of a quadratic form of rank $2n$ by a quadratic form of rank $4n$.

RÉSUMÉ. – Partant d'une forme modulaire parabolique de Hilbert de poids $2\kappa$, nous construisons une forme modulaire parabolique de Hilbert-Siegel de poids $\kappa + \frac{m}{2}$ et de degré m et son transfert aux formes intérieures des groupes symplectiques. Comme application, on obtient entre autres une relation entre les coefficients de Fourier de formes modulaires paraboliques de Hilbert de poids $n + \frac{1}{2}$ et une certaine somme pondérée des nombres de représentation d'une forme quadratique de rang $2n$ par une forme quadratique de rang $4n$.

## 1. Introduction

The present investigation deals with the following problem: starting from simple automorphic data such as cusp forms on $\mathrm{GL}_2$, construct more complicated automorphic forms on groups of higher degree. Toward this problem, Ikeda [22] has constructed a lifting associating to an elliptic cusp form a Siegel cusp form of even genus. This paper generalizes it to Hilbert cusp forms with different methods. The resulting Hilbert-Siegel cusp forms are applied to the theory of quadratic forms.

To illustrate our results, let $F$ be a totally real number field of degree $d$ with adèle ring $\mathbb{A}$. We write $\mathbb{A}_\mathbf{f}$ and $\mathbb{A}_\infty$ for the finite part and the infinite part of the adèle ring. We denote the set of $d$ real primes of $F$ by $\mathfrak{S}_\infty$ and the normalized absolute value by $\alpha = \prod_v \alpha_v : \mathbb{A}^\times \to \mathbb{R}_+^\times$.

Let $\mathrm{Sym}_m = \{z \in \mathrm{M}_m \mid {}^t z = z\}$ be the space of symmetric matrix of size $m$ and $W_m$ a symplectic vector space of dimension $2m$. We take matrix representation

$$\mathrm{Sp}_m = \left\{ g \in \mathrm{GL}_{2m} \,\middle|\, g \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}$$


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of the associated symplectic group $\mathrm{Sp}(W_m)$ by choosing a Witt basis of $W_m$. We define homomorphisms $\mathfrak{m} : \mathrm{GL}_m \to \mathrm{Sp}_m$ and $\mathfrak{n} : \mathrm{Sym}_m \to \mathrm{Sp}_m$ by

$$\mathfrak{m}(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \qquad \mathfrak{n}(b) = \begin{pmatrix} \mathbf{1}_m & b \\ 0 & \mathbf{1}_m \end{pmatrix}. \tag{1.1}$$

Let $\mathrm{Mp}(W_m)_{\mathbb{A}} \twoheadrightarrow \mathrm{Sp}_m(\mathbb{A})$ be the metaplectic double cover. Denote the inverse image of $\mathrm{Sp}_m(\mathbb{A}_\infty)$ (resp. $\mathrm{Sp}_m(\mathbb{A}_{\mathbf{f}})$) by $\mathrm{Mp}(W_m)_\infty$ (resp. $\mathrm{Mp}(W_m)_{\mathbf{f}}$).

Define the character $\mathbf{e}_\infty : \mathbb{C}^d \to \mathbb{C}^\times$ by $\mathbf{e}_\infty(\mathcal{Z}) = \prod_{v \in \mathfrak{S}_\infty} e^{2\pi\sqrt{-1}\mathcal{Z}_v}$. Let $\psi = \prod_v \psi_v$ be the additive character of $\mathbb{A}/F$ whose restriction to $\mathbb{A}_\infty$ is $\mathbf{e}_\infty|_{\mathbb{R}^d}$. We let $\mathfrak{p}$ denote a finite prime of $F$ and do not use $\mathfrak{p}$ for an archimedean place. For $\xi \in \mathrm{Sym}_m(F)$ we define the character $\psi^\xi_{\mathbf{f}} = \prod_{\mathfrak{p}} \psi^\xi_{\mathfrak{p}} : \mathrm{Sym}_m(\mathbb{A}_{\mathbf{f}}) \to \mathbb{C}^\times$ by $\psi^\xi_{\mathbf{f}}(z) = \prod_{\mathfrak{p}} \psi_{\mathfrak{p}}(\mathrm{tr}(\xi z_{\mathfrak{p}}))$. For $t \in F_v^\times$ there is an $8^{\text{th}}$ root of unity $\gamma(\psi^t_v)$ such that for all Schwartz functions $\phi$ on $F_v$

$$\int_{F_v} \phi(x_v)\psi_v(t x_v^2)\,\mathrm{d}x_v = \gamma(\psi^t_v)|2t|_v^{-1/2} \int_{F_v} \mathcal{F}\phi(x_v)\psi_v\left(-\frac{x_v^2}{4t}\right)\mathrm{d}x_v,$$

where $\mathrm{d}x_v$ is the self-dual Haar measure on $F_v$ with respect to the Fourier transform $\mathcal{F}\phi(y) = \int_{F_v} \phi(x_v)\psi_v(x_v y)\,\mathrm{d}x_v$. Set $\gamma^{\psi_v}(t) = \gamma(\psi_v)/\gamma(\psi^t_v)$. We denote the set of totally positive elements of $F$ by $F^\times_+$, the set of totally positive definite symmetric matrices of rank $m$ over $F$ by $\mathrm{Sym}^+_m$ and the set of all complex symmetric matrices of size $m$ with positive definite imaginary part by $\mathcal{H}_m$. For $t \in F^\times$ we write $\hat{\chi}^t = \prod_v \hat{\chi}^t_v$ for the quadratic character of $\mathbb{A}^\times/F^\times$ associated to the extension $F(\sqrt{t})/F$ and denote its restriction to the finite idèle group $\mathbb{A}^\times_{\mathbf{f}}$ by $\hat{\chi}^t_{\mathbf{f}}$. For $\ell \in \mathbb{R}^d$ we will set $|t|^\ell = \prod_{v \in \mathfrak{S}_\infty} |t|_v^{\ell_v}$.

The real metaplectic group $\mathrm{Mp}(W_m)_v$ acts on $\mathcal{H}_m$ through $\mathrm{Sp}_m(F_v)$ for $v \in \mathfrak{S}_\infty$. There is a unique factor of automorphy $\jmath : \mathrm{Mp}(W_m)_v \times \mathcal{H}_m \to \mathbb{C}^\times$ satisfying $\jmath(\tilde{g}_v, \mathcal{Z}_v)^2 = \det(C_v\mathcal{Z}_v + D_v)$. We here write the projection of $\tilde{g}_v$ to $\mathrm{Sp}_m(F_v)$ as $\begin{pmatrix} * & * \\ C_v & D_v \end{pmatrix}$. Let $\ell$ be a tuple of $d$ positive half integers such that $2\ell_v \equiv 2\ell_{v'} \pmod 2$ for all $v, v' \in \mathfrak{S}_\infty$. We set $J_\ell(\tilde{g}, \mathcal{Z}) = \prod_{v \in \mathfrak{S}_\infty} \jmath(\tilde{g}_v, \mathcal{Z}_v)^{2\ell_v}$ for $\tilde{g} \in \mathrm{Mp}(W_m)_\infty$ and $\mathcal{Z} \in \mathcal{H}^d_m$. If $\ell \in \mathbb{Z}^d$, then $J_\ell$ descends to the function on $\mathrm{Sp}_m(\mathbb{A}_\infty) \times \mathcal{H}^d_m$. Even when $\ell \notin \mathbb{Z}^d$, one can define it on some congruence subgroup $\Gamma^\theta_m$ of $\mathrm{Sp}_m(F)$. A Hilbert-Siegel modular form $\mathcal{F}$ of weight $\ell$ with respect to a congruence subgroup $\Gamma$ of $\mathrm{Sp}_m(F)$ is a holomorphic function on $\mathcal{H}^d_m$ which satisfies $\mathcal{F}(\gamma\mathcal{Z}) = \mathcal{F}(\mathcal{Z})J_\ell(\gamma, \mathcal{Z})$ for every $\gamma \in \Gamma$ and also the additional condition at infinity if $m = 1$ and $F = \mathbb{Q}$. Let $M_\ell(\Gamma)$ denote the vector space of such Hilbert-Siegel modular forms. The vector space $S_\ell(\Gamma)$ of Hilbert-Siegel cusp forms consists of functions $\mathcal{F} \in M_\ell(\Gamma)$ such that $\mathcal{F}(\gamma\mathcal{Z})\sqrt{J_{2\ell}(\gamma, \mathcal{Z})}^{-1}$ has a Fourier expansion of the form $\sum_{\xi \in \mathrm{Sym}^+_m} A(\xi)\mathbf{e}_\infty(\mathrm{tr}(\xi\mathcal{Z}))$ for all $\gamma \in \mathrm{Sp}_m(F)$, where $\sqrt{J_{2\ell}(\gamma, \mathcal{Z})}$ means any branch of the square root of $J_{2\ell}(\gamma, \mathcal{Z})$. Let $S^{(m)}_\ell$ denote the union of $S_\ell(\Gamma)$ for congruence subgroups $\Gamma \subset \Gamma^\theta_m$. The group $\mathrm{Mp}(W_m)_{\mathbf{f}}$ acts on the space $S^{(m)}_\ell$ and it is important to know which representations appear in this space. We shall explicitly construct a rather small irreducible submodule, which is neither tempered nor generic at any finite prime.

We define the space $\mathfrak{C}_{2\kappa}$ of Hilbert cusp forms on $\mathrm{PGL}_2$ of weight $2\kappa$ in Definition 5.4. Let $\pi_{\mathbf{f}} \simeq \otimes'_{\mathfrak{p}}\pi_{\mathfrak{p}}$ be an irreducible admissible unitary generic representation of $\mathrm{PGL}_2(\mathbb{A}_{\mathbf{f}})$. For some reason (see Remark 6.3) we suppose that none of $\pi_{\mathfrak{p}}$ is supercuspidal, i.e., there is a collection of continuous characters $\mu_{\mathfrak{p}}$ of the multiplicative groups of nonarchimedean local fields $F_{\mathfrak{p}}$ such that $\pi_{\mathfrak{f}}$ is equivalent to the unique irreducible submodule of the principal series representation $\bigotimes'_{\mathfrak{p}} I(\mu_{\mathfrak{p}}, \mu_{\mathfrak{p}}^{-1})$, where $\mu_{\mathfrak{p}}$ is unramified for almost all $\mathfrak{p}$. Put $\mu_{\mathfrak{f}} = \prod_{\mathfrak{p}} \mu_{\mathfrak{p}}$. We form the restricted tensor product $I_m^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}}) = \otimes'_{\mathfrak{p}} I_m^{\psi_{\mathfrak{p}}}(\mu_{\mathfrak{p}})$, where $I_m^{\psi_{\mathfrak{p}}}(\mu_{\mathfrak{p}})$ is the representation of the local metaplectic group $\mathrm{Mp}(W_m)_{\mathfrak{p}}$ on the space of smooth functions $h_{\mathfrak{p}}$ on $\mathrm{Mp}(W_m)_{\mathfrak{p}}$ transforming on the left according to

$$h_{\mathfrak{p}}((\mathfrak{m}(a)\mathfrak{n}(b), \zeta)\tilde{g}) = \zeta^m \gamma^{\psi_{\mathfrak{p}}}(\det a)^m \mu_{\mathfrak{p}}(\det a) |\det a|_{\mathfrak{p}}^{(m+1)/2} h_{\mathfrak{p}}(\tilde{g})$$

for all $\zeta \in \{\pm 1\}$, $a \in \mathrm{GL}_m(F_{\mathfrak{p}})$, $z \in \mathrm{Sym}_m(F_{\mathfrak{p}})$ and $\tilde{g} \in \mathrm{Mp}(W_m)_{\mathfrak{p}}$. This representation has a unique irreducible submodule $A_m^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}})$, which is unitary.

THEOREM 1.1. – *Notation being as above, $A_m^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}})$ appears in $S^{(m)}_{(2\kappa+m)/2}$ if and only if $\pi_{\mathfrak{f}}$ appears in $\mathfrak{C}_{2\kappa}$ and $(-1)^{\sum_{v \in \mathfrak{S}_\infty} \kappa_v} \prod_{\mathfrak{p}} \mu_{\mathfrak{p}}(-1) = 1$.*

We here denote the tuple $\left(\kappa_v + \frac{m}{2}\right)_{v \in \mathfrak{S}_\infty} \in \frac{1}{2}\mathbb{Z}^d$ simply by $(2\kappa + m)/2$. The representation $\pi_{\mathfrak{f}}$ is the Shimura correspondence of $A_1^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}})$ and $A_2^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}})$ is the Saito-Kurokawa lifting of $\pi_{\mathfrak{f}}$. Both are theta liftings. We discuss the connection of this result with Arthur's endoscopic classification in §6.2. Though the trace formula will ultimately lead to another proof, our proof, which relies heavily on the theory of the Shimura correspondence but not on the Saito-Kurokawa lifting, is completely elementary. If $\kappa_v < m$ for every $v \in \mathfrak{S}_\infty$, then we obtain an irreducible cuspidal automorphic representation which is nontempered at all the places.

More importantly, our proof gives more precise information. We can describe how the representation $A_m^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}})$ is embedded in $S^{(m)}_{(2\kappa+m)/2}$ quite explicitly. Fix a Haar measure $\mathrm{d}b = \bigotimes_{\mathfrak{p}} \mathrm{d}b_{\mathfrak{p}}$ on $\mathrm{Sym}_m(\mathbb{A}_{\mathfrak{f}})$. Then we can associate to each $\xi \in \mathrm{Sym}_m^+$ a basis vector $w_\xi^{\mu_{\mathfrak{f}}}$ of the one-dimensional vector space $\mathrm{Hom}_{\mathrm{Sym}_m(\mathbb{A}_{\mathfrak{f}})}(I_m^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}}) \circ \mathfrak{n}, \psi_{\mathfrak{f}}^\xi)$ by

$$w_\xi^{\mu_{\mathfrak{f}}}(\bigotimes_{\mathfrak{p}} h_{\mathfrak{p}}) = \prod_{\mathfrak{p}} w_\xi^{\mu_{\mathfrak{p}}}(h_{\mathfrak{p}}),$$

where $w_\xi^{\mu_{\mathfrak{p}}} \in \mathrm{Hom}_{\mathrm{Sym}_m(F_{\mathfrak{p}})}(I_m^{\psi_{\mathfrak{p}}}(\mu_{\mathfrak{p}}) \circ \mathfrak{n}, \psi_{\mathfrak{p}}^\xi)$ is defined by

$$\begin{aligned} w_\xi^{\mu_{\mathfrak{p}}}(h_{\mathfrak{p}}) = &\int_{\mathrm{Sym}_m(F_{\mathfrak{p}})} h_{\mathfrak{p}}\left(\left(\begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix} \mathfrak{n}(b_{\mathfrak{p}}), 1\right)\right) \overline{\psi_{\mathfrak{p}}^\xi(b_{\mathfrak{p}})}\, \mathrm{d}b_{\mathfrak{p}} \\ &\times \frac{|\det \xi|_{\mathfrak{p}}^{(m+1)/4}}{L\left(\frac{1}{2}, \mu_{\mathfrak{p}} \hat{\chi}_{\mathfrak{p}}^{\det \xi}\right)} \prod_{j=1}^{[(m+1)/2]} L(2j-1, \mu_{\mathfrak{p}}^2) \times \begin{cases} 1 & \text{if } 2 \nmid m, \\ L\left(\frac{m+1}{2}, \mu_{\mathfrak{p}} \hat{\chi}_{\mathfrak{p}}^{(-1)^{m/2}}\right) & \text{if } 2 | m. \end{cases} \end{aligned}$$

The integral diverges but makes sense as it stabilizes. One can check that $w_\xi^{\mu_{\mathfrak{p}}}(h_{\mathfrak{p}}) = 1$ for almost all $\mathfrak{p}$.

THEOREM 1.2. – *If $\pi_{\mathfrak{f}}$ appears in $\mathfrak{C}_{2\kappa}$ and $(-1)^{\sum_{v \in \mathfrak{S}_\infty} \kappa_v} \prod_{\mathfrak{p}} \mu_{\mathfrak{p}}(-1) = 1$, then $A_m^{\psi_{\mathfrak{f}}}(\mu_{\mathfrak{f}})$ appears in the decomposition of $S^{(m)}_{(2\kappa+m)/2}$ with multiplicity one, and there is a set $\{c_t\}_{t \in F_+^\times}$*