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BOUNDEDNESS OF $\mathbb{Q}$-FANO VARIETIES WITH DEGREES AND ALPHA-INVARIANTS BOUNDED FROM BELOW

BY CHEN JIANG

ABSTRACT. – We show that $\mathbb{Q}$-Fano varieties of fixed dimension with anti-canonical degrees and alpha-invariants bounded from below form a bounded family. As a corollary, K-semistable $\mathbb{Q}$-Fano varieties of fixed dimension with anti-canonical degrees bounded from below form a bounded family.

RÉSUMÉ. – Nous démontrons que les variétés de $\mathbb{Q}$-Fano de dimension fixe dont les degrés anticanoniques et les alpha-invariants sont bornés inférieurement forment une famille bornée. En corollaire, les variétés de $\mathbb{Q}$-Fano K-semistables de dimension fixe dont les degrés anticanoniques sont bornés inférieurement forment une famille bornée.

1. Introduction

Throughout the article, we work over an algebraically closed field of characteristic zero. A $\mathbb{Q}$-*Fano variety* is defined to be a normal projective variety X with at most klt singularities such that the anti-canonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

When the base field is the complex number field, an interesting problem for $\mathbb{Q}$-Fano varieties is the existence of Kähler-Einstein metrics which is related to K-(semi)stability of $\mathbb{Q}$-Fano varieties. It has been known that a *Fano manifold* X (i.e., a smooth $\mathbb{Q}$-Fano variety over $\mathbb{C}$) admits Kähler-Einstein metrics if and only if X is *K-polystable* by the works [15, 42, 16, 17, 14, 40, 32, 33, 3] and [11, 12, 13, 43]. K-stability is stronger than K-polystability, and K-polystability is stronger than K-semistability. Hence K-semistable $\mathbb{Q}$-Fano varieties are interesting for both differential geometers and algebraic geometers.

It also turned out that Kähler-Einstein metrics and K-stability play crucial roles for the construction of nice moduli spaces of certain $\mathbb{Q}$-Fano varieties. For example, compact moduli spaces of smoothable Kähler-Einstein $\mathbb{Q}$-Fano varieties have been constructed (see [36] for dimension two case and [30, 39, 34] for higher dimensional case). In order to consider the

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moduli space of certain (singular) $\mathbb{Q}$-Fano varieties, the first step is to show the boundedness property, which is the motivation of this paper. We show the boundedness of K-semistable $\mathbb{Q}$-Fano varieties of fixed dimension with anti-canonical degrees bounded from below, which gives an affirmative answer to a question asked by Yuchen Liu during the AIM workshop "Stability and moduli spaces" in January 2017.

THEOREM 1.1. – *Fix a positive integer d and a real number $\delta > 0$. Then the set of d-dimensional K-semistable $\mathbb{Q}$-Fano varieties X with $(-K_X)^d > \delta$ forms a bounded family.*

Note that the assumption that $(-K_X)^d$ is bounded from below is necessary, by Example 1.4(2) later.

As mentioned before, one might have further applications of Theorem 1.1 such as constructing moduli spaces of d-dimensional K-semistable $\mathbb{Q}$-Fano varieties with bounded anti-canonical degrees. An interesting corollary of Theorem 1.1 is the discreteness of the anti-canonical degrees of K-semistable $\mathbb{Q}$-Fano varieties.

COROLLARY 1.2. – *Fix a positive integer d. Then the set of $(-K_X)^d$ for d-dimensional K-semistable $\mathbb{Q}$-Fano varieties X is finite away from 0.*

Here a set $\mathscr{P}$ of positive real numbers is *finite away from* 0 if for any $\delta > 0$, $\mathscr{P} \cap (\delta, \infty)$ is a finite set. We remark that Corollary 1.2 might be related to the conjectural discreteness of minimal normalized volumes of klt singularities, cf. [31, Question 4.3].

The idea of proof of Theorem 1.1 comes from birational geometry. According to Minimal Model Program, $\mathbb{Q}$-Fano varieties form a fundamental class in birational geometry, and the boundedness property for $\mathbb{Q}$-Fano varieties is also interesting from the point view of birational geometry. For example, Kollár, Miyaoka, and Mori [26] proved that smooth Fano varieties form a bounded family. The most celebrated progress recently is the proof of Borisov-Alexeev-Borisov Conjecture due to Birkar [4, 5], which says that given a positive integer d and a real number $\epsilon > 0$, the set of ϵ-lc $\mathbb{Q}$-Fano varieties of dimension d forms a bounded family.

In this paper, inspired by Birkar's work, in order to show Theorem 1.1, we show the following theorem.

THEOREM 1.3. – *Fix a positive integer d and a real number $\delta > 0$. Then the set of d-dimensional $\mathbb{Q}$-Fano varieties X with $(-K_X)^d > \delta$ and $\alpha(X) > \delta$ forms a bounded family.*

Here $\alpha(X)$ is the *alpha-invariant* of X defined by Tian [41] (see also [7]) in order to investigate the existence of Kähler-Einstein metrics on Fano manifolds. Recall that Fujita and Odaka [18, Theorem 3.5] proved that the alpha-invariant of a K-semistable $\mathbb{Q}$-Fano variety of dimension d is always not less than $1/(d+1)$, so Theorem 1.3 implies Theorem 1.1 naturally. The advantage to consider Theorem 1.3 is that we can then apply methods from birational geometry, instead of dealing with K-semistable $\mathbb{Q}$-Fano varieties.

The point of Theorem 1.3 is that we replace the ϵ-lc condition in Borisov-Alexeev-Borisov Conjecture by the condition on lower bound of anti-canonical degrees and alpha-invariants, which are global invariants.

We remark that if one takes $\delta = 1$, then Theorem 1.3 is a consequence of [4, Theorem 1.3], which says that the set of *exceptional* $\mathbb{Q}$-Fano varieties (i.e., $\mathbb{Q}$-Fano varieties X with $\alpha(X) > 1$) of fixed dimension forms a bounded family. Note that in this case we do not even need to assume $(-K_X)^d$ is bounded from below. But in general we need to assume both $(-K_X)^d$ and $\alpha(X)$ are bounded from below, by the following examples.

EXAMPLE 1.4. – Fix a positive integer d.

1. Consider the weighted projective space $X_n = \mathbb{P}(1^d, n)$ which is a $\mathbb{Q}$-Fano variety of dimension d with $(-K_{X_n})^d = (n+d)^d/n > 1$, but it is clear that $\{X_n\}$ does not form a bounded family.
2. Consider $Y_{8n+4} \subset \mathbb{P}(2, 2n+1, 2n+1, 4n+1)$, a general weighted hypersurface of degree $8n+4$, which is a $\mathbb{Q}$-Fano variety of dimension 2 with $\alpha(Y_{8n+4}) = 1$ (see [6, Corollary 1.12] or [22]), but it is clear that $\{Y_{8n+4}\}$ does not form a bounded family. For more interesting examples of $\mathbb{Q}$-Fano varieties with $\alpha \geq 1$, we refer to [6, 9] in dimension 2 and [8, 10] in higher dimensions. Note that all examples with $\alpha \geq 1$ are K-semistable (in fact, K-stable) by [35, Theorem 1.4] (or [41]).

By [4, Proposition 7.13] or [5, Theorem 2.15], Theorem 1.3 is a consequence of the following theorem.

THEOREM 1.5. – *Fix a positive integer d and a real number $\delta > 0$. Then there exists a positive integer m depending only on d and δ such that if X is a d-dimensional $\mathbb{Q}$-Fano variety with $(-K_X)^d > \delta$ and $\alpha(X) > \delta$, then $|-mK_X|$ defines a birational map.*

To show Theorem 1.5, our main idea is to establish an inequality expressed in terms of the volume of $-K_X|_G$ on a covering family of subvarieties G of X and $(-K_X)^d$, $\alpha(X)$, see Lemma 3.1.

As a variation of Theorem 1.3, we can also show the following theorem.

THEOREM 1.6. – *Fix a positive integer d and a real number $\theta > 0$. Then the set of d-dimensional $\mathbb{Q}$-Fano varieties X with $\alpha(X)^d \cdot (-K_X)^d > \theta$ forms a bounded family.*

Logically, Theorem 1.3 is implied by Theorem 1.6. But we will show Theorem 1.3 first in order to make the explanation more clear.

REMARK 1.7. – Note that the invariant $\alpha(X)^d \cdot (-K_X)^d$ appears naturally in birational geometry, see for example [25, Theorem 6.7.1]. It is not clear whether we can replace $\alpha(X)^d \cdot (-K_X)^d$ in Theorem 1.6 by $\alpha(X)^{d'} \cdot (-K_X)^d$ for some positive real number $d' < d$. At least $d' \leq d-1$ is not sufficient to conclude the boundedness. For example, in Example 1.4(1), $(-K_{X_n})^d = (n+d)^d/n$ and $\alpha(X_n) = 1/(n+d)$ (for computation of alpha-invariants of toric varieties, see [1, 6.3]), hence $\alpha(X_n)^{d-1} \cdot (-K_{X_n})^d > 1$.

REMARK 1.8. – We remark that the proof of both Theorems 1.3 and 1.6 works under the weaker assumption that X is a *weak $\mathbb{Q}$-Fano variety* (i.e., X has at most klt singularities and $-K_X$ is nef and big), see also Remark 2.5. But it is not clear yet whether the log Fano pair versions hold or not.