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VIRTUALLY TORSION-FREE COVERS OF MINIMAX GROUPS

BY PETER KROPHOLLER AND KARL LORENSEN

ABSTRACT. – We prove that every finitely generated, virtually solvable minimax group can be expressed as a homomorphic image of a virtually torsion-free, virtually solvable minimax group. This result enables us to generalize a theorem of Ch. Pittet and L. Saloff-Coste about random walks on finitely generated, virtually solvable minimax groups. Moreover, the paper identifies properties, such as the derived length and the nilpotency class of the Fitting subgroup, that are preserved in the covering process. Finally, we determine exactly which infinitely generated, virtually solvable minimax groups also possess this type of cover.

RÉSUMÉ. – Nous prouvons que tout groupe de type fini minimax et virtuellement résoluble peut être exprimé comme l'image homomorphe d'un groupe minimax virtuellement résoluble et virtuellement sans torsion. Ce résultat permet de généraliser un théorème de Ch. Pittet et L. Saloff-Coste concernant les marches aléatoires sur les groupes de type fini minimax et virtuellement résolubles. En outre, l'article identifie des propriétés conservées dans le processus de couverture, telles que la classe de résolubilité et la classe de nilpotence du sous-groupe de Fitting. Enfin, nous déterminons exactement quels groupes de type infini minimax et virtuellement résolubles admettent également ce type de couverture.

1. Introduction

In this paper, we study *virtually solvable minimax groups*; these are groups G that possess a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G,$$

such that each factor G_i/G_{i-1} is either finite, infinite cyclic, or quasicyclic. (Recall that a group is *quasicyclic* if, for some prime p, it is isomorphic to $\mathbb{Z}(p^\infty) := \mathbb{Z}[1/p]/\mathbb{Z}$.) Denoting

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the class of virtually solvable minimax groups by $\mathfrak{M}$, we determine which $\mathfrak{M}$-groups can be realized as quotients of virtually torsion-free $\mathfrak{M}$-groups. Moreover, our results on quotients allow us to settle a longstanding question about a lower bound for the return probability of a random walk on the Cayley graph of a finitely generated $\mathfrak{M}$-group (see § 1.3). We are indebted to Lison Jacoboni for pointing out the relevance of our work to this question.

The importance of $\mathfrak{M}$-groups arises primarily from the special status among virtually solvable groups that is enjoyed by finitely generated $\mathfrak{M}$-groups. As shown by the first author in [14], the latter comprise all the finitely generated, virtually solvable groups without any sections isomorphic to a wreath product of a finite cyclic group with an infinite cyclic one. In particular, any finitely generated, virtually solvable group of finite abelian section rank is minimax (a property first established by D. J. S. Robinson [23]). For background on these and other properties of $\mathfrak{M}$-groups, we refer the reader to J. C. Lennox and Robinson's treatise [16] on infinite solvable groups.

Within the class $\mathfrak{M}$, we distinguish two subclasses: first, the subclass $\mathfrak{M}_1$ consisting of all the $\mathfrak{M}$-groups that are virtually torsion-free; second, the complement of $\mathfrak{M}_1$ in $\mathfrak{M}$, which we denote $\mathfrak{M}_\infty$. It has long been apparent that the groups in $\mathfrak{M}_1$ possess a far more transparent structure than those in $\mathfrak{M}_\infty$. For example, an $\mathfrak{M}$-group belongs to $\mathfrak{M}_1$ if and only if it is residually finite. As a result, every finitely generated $\mathfrak{M}_\infty$-group fails to be linear over any field. In contrast, $\mathfrak{M}_1$-groups are all linear over $\mathbb{Q}$ and hence can be studied with the aid of the entire arsenal of the theory of linear groups over $\mathbb{R}$, including via embeddings into Lie groups. The latter approach is particularly fruitful when tackling problems of an analytic nature, such as those that arise in the investigation of random walks (see [21]).

A further consequence of the $\mathbb{Q}$-linearity of $\mathfrak{M}_1$-groups is that the finitely generated ones fall into merely countably many isomorphism classes. On the other hand, there are uncountably many nonisomorphic, finitely generated $\mathfrak{M}_\infty$-groups (see either [16, p. 104] or Proposition 7.12 below). Other differences between $\mathfrak{M}_1$-groups and $\mathfrak{M}_\infty$-groups are evident in their respective algorithmic properties. The word problem, for instance, is solvable for every finitely generated group in $\mathfrak{M}_1$ (see [2]). However, since the number of possible algorithms is countable, there exist uncountably many nonisomorphic, finitely generated $\mathfrak{M}_\infty$-groups with unsolvable word problem.

Our goal here is to explore the following two questions concerning the relationship between $\mathfrak{M}_1$-groups and $\mathfrak{M}_\infty$-groups. In phrasing the questions, we employ a parlance that will be used throughout the paper, and that also occurs in its title: if a group G can be expressed as a homomorphic image of a group G^*, we say that G is *covered by* G^*. In this case, G^* is referred to as a *cover* of G and any epimorphism $G^* \to G$ as a *covering*. A cover of G that belongs to the class $\mathfrak{M}_1$ is called an $\mathfrak{M}_1$-*cover* of G, and the corresponding covering is designated an $\mathfrak{M}_1$-*covering*.

QUESTION 1.1. – *Under what conditions is it possible to cover an* $\mathfrak{M}_\infty$*-group by an* $\mathfrak{M}_1$*-group?*

QUESTION 1.2. – *If an* $\mathfrak{M}_\infty$*-group* G *can be covered by an* $\mathfrak{M}_1$*-group* G^**, how can we choose* G^* *so that it retains many of the properties enjoyed by* G*?*

Answering the above questions should enable us to reduce certain problems about $\mathfrak{M}$-groups to the more tractable case where the group belongs to $\mathfrak{M}_1$. A current problem inviting such an approach arises in Ch. Pittet and L. Saloff-Coste's study [21] of random walks on the Cayley graphs of finitely generated $\mathfrak{M}$-groups. Their paper establishes a lower bound on the probability of return for this sort of random walk when the group is virtually torsion-free, but their methods fail to apply to $\mathfrak{M}_\infty$-groups.(1) One way to extend their bound to the latter case is to prove that any finitely generated member of the class $\mathfrak{M}_\infty$ can be expressed as a homomorphic image of an $\mathfrak{M}_1$-group. In Theorem 1.5 below, we establish that the condition that the group be finitely generated is indeed one possible answer to Question 1.1, thus yielding the desired generalization of Pittet and Saloff-Coste's result (Corollary 1.8).

1.1. Structure of $\mathfrak{M}$-groups

Before stating our main results, we summarize the structural properties of $\mathfrak{M}$-groups in Proposition 1.3 below; proofs of these may be found in [16]. In the statement of the proposition, as well as throughout the rest of the paper, we write $\mathrm{Fitt}(G)$ for the *Fitting subgroup* of a group G, namely, the subgroup generated by all the nilpotent normal subgroups. In addition, we define $R(G)$ to be the *finite residual* of G, by which we mean the intersection of all the subgroups of finite index.

PROPOSITION 1.3. – *If G is an $\mathfrak{M}$-group, then the following four statements hold.*

(i) $\mathrm{Fitt}(G)$ *is nilpotent and* $G/\mathrm{Fitt}(G)$ *virtually abelian.*

(ii) *If G belongs to $\mathfrak{M}_1$, then $G/\mathrm{Fitt}(G)$ is finitely generated.*

(iii) *$R(G)$ is a direct product of finitely many quasicyclic groups.*

(iv) *G is a member of $\mathfrak{M}_1$ if and only if $R(G) = 1$.*

1.2. Covers of finitely generated $\mathfrak{M}$-groups

Our main result, Theorem 1.13, characterizes all the $\mathfrak{M}$-groups that can be covered by an $\mathfrak{M}_1$-group. We will postpone describing this theorem until § 1.4, focusing first on its implications for finitely generated $\mathfrak{M}$-groups. We begin with this special case because of its immediate relevance to random walks, as well as its potential to find further applications.

While discussing our results, we will refer to the following example; it is the simplest instance of a finitely generated $\mathfrak{M}_\infty$-group, originally due to P. Hall [7]. More sophisticated examples of such groups are described in the final section of the paper.

EXAMPLE 1.4. – Take p to be a prime, and let G^* be the group of upper triangular 3×3 matrices (a_{ij}) with entries in the ring $\mathbb{Z}[1/p]$ such that $a_{11} = a_{33} = 1$ and a_{22} is a power of p. Let A be the central subgroup consisting of all the matrices $(a_{ij}) \in G^*$ with $a_{22} = 1$, $a_{12} = a_{23} = 0$, and $a_{13} \in \mathbb{Z}$. Set $G = G^*/A$. Then G is a finitely generated solvable minimax group with a quasicyclic center.

(1) The hypothesis that the group is virtually torsion-free should be included in the statement of [21, Theorem 1.1], for the proof requires that assumption. We thank Lison Jacoboni for bringing this mistake to our attention.