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Luis A. FLORIT & Wolfgang ZILLER

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

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MANIFOLDS WITH CONULLITY AT MOST TWO AS GRAPH MANIFOLDS

BY LUIS A. FLORIT AND WOLFGANG ZILLER

ABSTRACT. – We find necessary and sufficient conditions for a complete Riemannian manifold M^n of finite volume, whose curvature tensor has nullity at least $n-2$, to be a geometric graph manifold. In the process, we show that Nomizu's conjecture, well known to be false in general, is true for manifolds with finite volume.

RÉSUMÉ. – Nous trouvons les conditions nécessaires et suffisantes pour qu'une variété riemannienne complète M^n de volume fini, dont le tenseur de courbure a nullité au moins $n-2$, soit une variété graphe géométrique. Dans le processus, nous montrons que la conjecture de Nomizu, bien connue pour être fausse en général, est vraie pour les variétés à volume fini.

The nullity space Γ of the curvature tensor R of a Riemannian manifold M^n is defined for each $p \in M$ as $\Gamma(p) = \{X \in T_pM : R(X, Y) = 0 \ \ \forall Y \in T_pM\}$, and its dimension $\mu(p)$ is called the *nullity* of M^n at p. It is well known that the existence of points with positive nullity has strong geometric implications. For example, on an open subset of M^n where μ is constant, Γ is an integrable distribution with totally geodesic leaves. In addition, if M^n is complete, its leaves are also complete on the open subset where μ is minimal; see e.g., [12]. Riemannian n-manifolds with conullity at most 2, i.e., $\mu \geq n-2$, which we call *CN2 manifolds* for short, appear naturally and frequently in several different contexts in Riemannian geometry, e.g.,:

a) Gromov's 3-dimensional graph manifolds admit a complete CN2 metric with nonpositive sectional curvature and finite volume whose set of flat points consists of a disjoint union of flat totally geodesic tori ([11]). These were the first examples of Riemannian manifolds with geometric rank one. Interestingly, any complete metric of nonpositive curvature on such a graph manifold is necessarily CN2 and quite rigid, as was shown in [17].

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b) A Riemannian manifold is called semi-symmetric if at each point the curvature tensor is orthogonally equivalent to the curvature tensor of some symmetric space, which is allowed to depend on the point. CN2 manifolds are semi-symmetric since they have pointwise the curvature tensor of an isometric product of a Euclidean space and a surface with constant curvature. Conversely, Szabó showed in [22] that a complete simply connected semi-symmetric space is isometric to a Riemannian product $S \times N$, where S is a symmetric space and N is, on an open and dense subset, locally a product of CN2 manifolds.

c) Isometrically deformable submanifolds tend to have large nullity. In particular, by the classic Beez-Killing theorem, any locally deformable hypersurface in a space form has to be CN2. Yet, generically, CN2 hypersurfaces are locally rigid, and the classification of the deformable ones has been carried out a century ago in [15, 4]; see [7] for a modern version and further results. The corresponding classification of locally deformable CN2 Euclidean submanifolds in codimension two is considerably more involved, and was obtained only recently in [6] and [8].

d) A compact immersed submanifold $M^3 \subset \mathbb{R}^5$ with nonnegative sectional curvature not diffeomorphic to the 3-sphere $\mathbb{S}^3$ is necessarily CN2, and either isometric to $(\mathbb{S}^2 \times \mathbb{R})/\mathbb{Z}$ for some metric of nonnegative curvature on $\mathbb{S}^2$, or diffeomorphic to a lens space $\mathbb{S}^3/\mathbb{Z}_p$; see [9]. In the case of lens spaces, the set of points with vanishing curvature has to be nonempty with Hausdorff dimension at least two. However, it is not known yet if they can be isometrically immersed into $\mathbb{R}^5$.

e) I. M. Singer asked in [21] whether a Riemannian manifold is homogeneous if the curvature tensor at any two points is orthogonally equivalent. The first counterexamples to this question were CN2 manifolds with constant scalar curvature, which clearly have this property, and are typically not homogeneous; see [18, 3].

The most trivial class of CN2 manifolds is given by cylinders $L^2 \times \mathbb{R}^{n-2}$ with their natural product metrics, where L^2 is any (not necessarily complete) connected surface. More generally, we call a *twisted cylinder* any quotient

$$C^n = (L^2 \times \mathbb{R}^{n-2})/G,$$

where $G \subset \mathrm{Iso}(L^2 \times \mathbb{R}^{n-2})$ acts properly discontinuously and freely. The natural quotient metric is clearly CN2, and we call L^2 the *generating surface* of C^n, and the images of the Euclidean factor its *nullity leaves*. Observe that C^n fails to be complete only because L^2 does not need to be. Yet, what is important for us is that C^n is foliated by complete, flat, totally geodesic, and locally parallel leaves of codimension 2.

Our first goal is to show that these are the basic building blocks of complete CN2 manifolds with finite volume:

THEOREM A. – *Let M^n be a complete CN2 manifold. Then each finite volume connected component of the set of nonflat points of M^n is globally isometric to a twisted cylinder.*

The hypothesis on the volume of M^n is essential, since complete locally irreducible Riemannian manifolds with constant conullity two abound in any dimension; see [18, 3] and references therein. These examples serve also as counterexamples to the Nomizu conjecture in [13], which states that a complete locally irreducible semi-symmetric space of dimension at least three must be locally symmetric. However, Theorem A together with Theorem 4.4 in [22] yield:

Corollary 1. – *Nomizu's conjecture is true for manifolds with finite volume.*

For the 3-dimensional case, the fact that the set of nonflat points of a finite volume CN2 manifold is locally reducible was proved in [19] and [16] with a longer and more delicate proof; see also [20] for the 4-dimensional case. Notice also that in dimension 3 the CN2 condition is equivalent to the assumption, called cvc(0) in [16], that every tangent vector is contained in a flat plane, or to the condition that the Ricci endomorphism has eigenvalues $(\lambda, \lambda, 0)$. Furthermore, in [2] it was shown that a complete 3-manifold with (geometric) rank one is a twisted cylinder.

Observe that we are free to change the metric in the interior of the generating surfaces of the twisted cylinders in Theorem A, still obtaining a complete CN2 manifold. Moreover, they are nowhere flat with Gaussian curvature vanishing at their boundaries. Of course, these boundaries can be quite complicated and irregular.

In general it is very difficult to fully understand how the twisted cylinders in Theorem A can be glued together through the set of flat points in order to build a complete Riemannian manifold. An obvious way of gluing them is through compact totally geodesic flat hypersurfaces. Indeed, when the boundary of each generating surface L^2 in the twisted cylinder $C = (L^2 \times \mathbb{R}^{n-2})/G$ is a disjoint union of complete geodesics γ_j along which the Gaussian curvature of L^2 vanishes to infinity order, the boundary of C is a disjoint union of complete totally geodesic flat hypersurfaces $H_j = (\gamma_j \times \mathbb{R}^{n-2})/G_j \subset M^n$, where G_j is the subgroup of G preserving γ_j. We can now use each H_j to attach another finite volume twisted cylinder C' to C along H_j, as long as C' has a boundary component isometric to H_j. Repeating and iterating this procedure with each boundary component we construct a complete CN2 manifold M^n. As we will see, the hypersurfaces H_j have to be compact if M^n has finite volume. This motivates the following concept of geometric graph manifold of dimension $n \geq 3$, which by definition is endowed with a CN2 Riemannian metric:

Definition 1. – *A connected Riemannian manifold M^n is called a geometric graph manifold if M^n is a locally finite disjoint union of twisted cylinders C_i glued together through disjoint compact totally geodesic flat hypersurfaces H_λ of M^n. That is,*

$$M^n \setminus W = \bigsqcup_\lambda H_\lambda, \quad \textit{where} \quad W := \bigsqcup_i C_i.$$

Here we allow the possibility that a hypersurface H_λ is one-sided, even when M^n is orientable. We also assume, without loss of generality, that the nullity leaves of two cylinders C and C', glued along H_λ, have distinct limits in H_λ. This implies in particular that for each cylinder C, the Gauss curvature vanishes along ∂C to infinite order. Notice that the locally finiteness condition is equivalent to the assumption that each H_λ is a common