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Daping WENG

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

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DONALDSON-THOMAS TRANSFORMATION OF DOUBLE BRUHAT CELLS IN SEMISIMPLE LIE GROUPS

BY DAPING WENG

ABSTRACT. – Double Bruhat cells $G^{u,v}$ were first studied by Fomin and Zelevinsky [8]. They provide important examples of cluster algebras [1] and cluster Poisson varieties [5]. Cluster varieties produce examples of 3d Calabi-Yau categories with stability conditions, and their Donaldson-Thomas invariants, defined by Kontsevich and Soibelman [15], are encoded by a formal automorphism on the cluster variety known as the Donaldson-Thomas transformation. Goncharov and Shen conjectured in [11] that for any semisimple Lie group G, the Donaldson-Thomas transformation of the cluster Poisson variety $H\backslash G^{u,v}/H$ is a slight modification of Fomin and Zelevinsky's twist map [8]. In this paper we prove this conjecture, using crucially Fock and Goncharov's cluster ensembles [7] and the amalgamation construction [5]. Our result, combined with the work of Gross, Hacking, Keel, and Kontsevich [12], proves the duality conjecture of Fock and Goncharov [7] in the case of $H\backslash G^{u,v}/H$.

RÉSUMÉ. – Les doubles cellules de Bruhat $G^{u,v}$ ont initialement été étudiées par Fomin et Zelevinsky [8]. Ils fournissent des exemples importants d'algèbres amassées [1] et de variétés de Poisson amassées [5]. Les variétés amassées produisent des exemples de categories de 3d Calabi-Yau satisfaisant certaines conditions de stabilité; et leurs invariants de Donaldson-Thomas définis par Kontsevich et Soibelman [15] sont encodés par un automorphisme formel sur la variété amassée connue sous le nom de transformation de Donaldson-Thomas. Goncharov et Shen conjecturent dans [11] que pour tout groupe de Lie semi-simple G, la transformation de Donaldson-Thomas de la variété de Poisson amassée $H\backslash G^{u,v}/H$ est une modification mineure de l'application twist de Fomin et Zelevinsky [8]. Dans cet article, nous démontrons cette conjecture en utilisant les ensembles amassés et la construction d'amalgames de Fock et Goncharov de façon essentielle. Notre résultat, une fois combiné avec les travaux de Gross, Hacking, Keel, et Kontsevich, démontre la conjecture de dualité de Fock et Goncharov dans le cas de $H\backslash G^{u,v}/H$.

1. Introduction

Cluster algebras were defined by Fomin and Zelevinsky in [9]. Cluster varieties were introduced by Fock and Goncharov in [7]. They can be used to construct examples of 3d Calabi-Yau categories with stability conditions. One important object of study for such categories are their Donaldson-Thomas invariants, introduced by Kontsevich and Soibelman [15], which generalize geometric invariants of Calabi-Yau manifolds. For a 3d Calabi-Yau

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category with stability condition constructed from a quiver (seed) together with a generic potential, its Donaldson-Thomas invariants are encoded by a single formal automorphism on the corresponding cluster variety, which is also known as the Donaldson-Thomas transformation [15]. Keller [14] gave a combinatorial characterization of a certain class of Donaldson-Thomas transformations based on quiver mutation. Goncharov and Shen gave an equivalent definition of the Donaldson-Thomas transformations using tropical points of cluster varieties in [11], which we will use in this paper.

Double Bruhat cells form an important family of examples in the study of cluster algebras and cluster Poisson varieties since the very beginning of the subject. On the one hand, Berenstein, Fomin, and Zelevinsky [1] proved that the algebra of regular functions on double Bruhat cells in simply connected semisimple Lie groups are upper cluster algebras. On the other hand, Fock and Goncharov [5] showed that double Bruhat cells in adjoint semisimple Lie groups are cluster Poisson varieties. Furthermore, Fock and Goncharov proved in the same paper that the Poisson structure in the biggest double Bruhat cell, which is a Zariski open subset of the Lie group, coincides with the Poisson-Lie structure defined by Drinfeld in [4]. Williams [20] showed that these two constructions can be combined into a cluster ensemble in the sense of Fock and Goncharov [7], which will play a central role in our construction of Donaldson-Thomas transformation on the double quotient $H\backslash G^{u,v}/H$.

Recall the flag variety $\mathscr{B}$ associated to a semisimple Lie group G. Generic configurations of flags were studied by Fock and Goncharov in [6]. The cluster Donaldson-Thomas transformation of such configuration space was constructed by Goncharov and Shen in [11]. In this paper we make use of the configuration space of quadruple of flags with certain degenerate conditions depending on a pair of Weyl group elements (u, v), which we call $\mathrm{Conf}^{u,v}(\mathscr{B})$, and show that such configuration space is isomorphic to the quotient $H\backslash G^{u,v}/H$ of double Bruhat cells.

Our main result is to show that the Donaldson-Thomas transformation on the cluster Poisson variety associated to $H\backslash G^{u,v}/H$ is equivalent to a modified version of Fomin and Zelevinsky's twist map [8], which is in turn equivalent to some explicit automorphism on the configuration space $\mathrm{Conf}^{u,v}(\mathscr{B})$ as well. Related versions of the twist map in the case of Grassmannian have been studied independently by Marsh and Scott [16], by Muller and Speyer [17], and by the author [19].

1.1. Main Result

To state our main result, we need to first introduce the related spaces and maps between them.

Let G be a semisimple Lie group. Fix a pair of opposite Borel subgroups $B_\pm$ of G and let $H := B_+ \cap B_-$ be the corresponding maximal torus. Then with respect to these two Borel subgroups, G admits two Bruhat decompositions

$$G = \bigsqcup_{w\in W} B_+ w B_+ = \bigsqcup_{w\in W} B_- w B_-,$$

where W denotes the Weyl group with respect to the maximal torus H. For a pair of Weyl group elements (u, v), we define the *double Bruhat cell* $G^{u,v}$ to be the intersection

$$G^{u,v} := B_+ u B_+ \cap B_- v B_-.$$

One of the most important map in the study of double Bruhat cells is Fomin and Zelevinsky's twist map, which they introduced in [8]

$$\begin{aligned} \mathrm{tw} : G^{u,v} &\to G^{u^{-1},v^{-1}} \\ x &\mapsto \left(\left[\overline{u}^{-1}x\right]_-^{-1} \overline{u}^{-1} x \overline{v^{-1}} \left[x\overline{v^{-1}}\right]_+^{-1} \right)^{t\circ\iota}. \end{aligned}$$

Here $x = [x]_-[x]_0[x]_+$ denotes the Gaussian decomposition, $\overline{w}$ denotes the lift of a Weyl group element w defined by Equation (2.9) (see also [8], [1], and [11]), and ι is anti-involution on G that maps $G^{u,v}$ biregularly to $G^{u^{-1},v^{-1}}$ (see Equation (2.5)).

Double Bruhat cells are closely related to cluster varieties. On the one hand, Berenstein, Fomin, and Zelevinsky [1] showed that when G is a simply connected semisimple Lie group (which we will indicate by a subscript sc in this paper), the coordinate ring $\mathcal{O}\left(G_{\mathrm{sc}}^{u,v}\right)$ has the structure of a cluster algebra, which can be interpreted as a birational equivalence with a cluster $\mathcal{A}$-variety $\mathcal{A}^{u,v}$

$$\psi : G_{\mathrm{sc}}^{u,v} \dashrightarrow \mathcal{A}^{u,v}.$$

On the other hand, Fock and Goncharov [5] defined amalgamation map for double Bruhat cells in an adjoint semisimple Lie group, which induces a birational equivalence between the cluster $\mathcal{X}$-variety $\mathcal{X}^{u,v}$ corresponding to $\mathcal{A}^{u,v}$ and the double quotient $H\backslash G^{u,v}/H$

$$\chi : \mathcal{X}^{u,v} \to H\backslash G/H.$$

On the cluster Poisson variety $\mathcal{X}^{u,v}$, one question we can ask is whether it possesses a Donaldson-Thomas transformation DT in the sense of Goncharov and Shen [11], and if yes, a follow-up question to ask is whether the Donaldson-Thomas transformation is a cluster transformation. Our main result provides positive answers to both questions.

The next space we would like to introduce is the configuration space of quadruple of flags. Let $\mathcal{B}$ be the flag variety associated to the semisimple Lie group G. By using the Borel subgroup B_+ we can define a bijection between the G-orbits in $\mathcal{B} \times \mathcal{B}$ and the Weyl group W; in particular, it is known that two Borel subgroups B_1 and B_2 are opposite if and only if the pair (B_1, B_2) is in the orbit corresponding to the longest Weyl group element w_0. For any two Borel subgroups B_1 and B_2, we write $B_1 \xrightarrow{w} B_2$ if the pair (B_1, B_2) is in the orbit corresponding to the Weyl group element w, and write $B \text{———} B'$ if B and B' are opposite Borel subgroups. Then we define the configuration space $\mathrm{Conf}^{u,v}(\mathcal{B})$ to be the configuration space of quadruple of Borel subgroups (B_1, B_2, B_3, B_4) satisfying the relative condition

$$\begin{tikzcd} B_1 \arrow[r, "u"] \arrow[d, dash] & B_2 \arrow[d, dash] \\ B_3 \arrow[r, "v^*"'] & B_4, \end{tikzcd}$$

with $v^* := w_0 v w_0$, modulo the diagonal adjoint action by G. As it turns out, $\mathrm{Conf}^{u,v}(\mathcal{B})$ is naturally isomorphic to the double quotient of the double Bruhat cell $H\backslash G^{u,v}/H$ (Proposition 2.28).