ISSN 0012-9593

quatrième série - tome 53 fascicule 3 mai-juin 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 53, 2020, p. 787 à 814

ON THE DIVERIO-TRAPANI CONJECTURE

BY YA DENG

ABSTRACT. – In this paper we establish effective lower bounds on the degrees of the Debarre and Kobayashi conjectures. Then we study a more general conjecture proposed by Diverio-Trapani on the ampleness of jet bundles of general complete intersections in complex projective spaces.

RÉSUMÉ. – Dans cet article, nous établissons des bornes inférieures effectives sur les degrés liés aux conjectures de Debarre et Kobayashi. Ensuite, nous étudions une conjecture plus générale proposée par Diverio-Trapani sur l'amplitude des fibrés de jets des intersections complètes générales dans les espaces projectifs complexes.

0. Introduction

A compact complex manifold X is said to be Kobayashi (Brody) hyperbolic if there exists no non-constant holomorphic map $f : \mathbb{C} \to X$. As is well-known, a sufficient criteria for Kobayashi hyperbolicity is the ampleness of the cotangent bundle. Although the complex manifolds with ample cotangent bundles are expected to be reasonably abundant, there are few concrete constructions before the work of Debarre. In [8], Debarre proved that the complete intersection of sufficiently ample general hypersurfaces in a *complex abelian variety*, whose codimension is at least as large as its dimension, has ample cotangent bundle. He further conjectured that this result should also hold for intersection varieties of general hypersurfaces in *complex projective spaces* (the so-called *Debarre conjecture*). This conjecture was recently proved by Brotbek-Darondeau [4] and independently by Xie [26, 25], based on the ideas and explicit methods in [2].

THEOREM 0.1 (Brotbek-Darondeau, Xie). – *Let X be an n-dimensional projective manifold equipped with a very ample line bundle $\mathscr{A}$. Then there exists $d_{\mathrm{Deb},n} \in \mathbb{N}$ depending only on the dimension n, such that for all $d \geqslant d_{\mathrm{Deb},n}$, the complete intersection of c-general hypersurfaces $H_1, \ldots, H_c \in |\mathscr{A}^d|$ has ample cotangent bundle, provided that $\frac{n}{2} \leqslant c \leqslant n$.*

 doi:10.24033/asens.2434

In [25], Xie was able to obtain an effective lower bound $d_{\mathrm{Deb},n} = n^{n^2}$ by working with (much more elaborated) explicit expressions of some symmetric differential forms. The result in [4] is "almost" effective on $d_{\mathrm{Deb},n}$, because it depends on some constant involved in some noetherianity argument, arising in their reduction to Nakamaye's theorem [22] for *families of zero-dimensional subschemes*.

One goal of the present paper is to provide an effective estimate for such a Nakamaye's theorem (see Theorem 2.10). In particular, as a complement of [4, Theorem 1.1], we can improve Xie's effective lower bound $d_{\mathrm{Deb},n}$.

THEOREM A. – *In the same setting as Theorem 0.1, one can take*

$$d_{\mathrm{Deb},n} = (2n)^{n+3}.$$

It is worth to mention that the techniques in [4] are more intrinsic and the ideas of their proof brought new geometric insights in the understanding of the positivity of cotangent bundles. Later, Brotbek [3] extended these techniques from the setting of symmetric differentials to that of higher order jet differentials, so that he was able to prove a long-standing conjecture of Kobayashi in [19].

THEOREM 0.2 (Brotbek). – *Let X be a projective manifold of dimension n. For any very ample line bundle $\mathscr{A}$ on X, there exists $d_{\mathrm{Kob},n} \in \mathbb{N}$ depending only on the dimension n such that for any $d \geqslant d_{\mathrm{Kob},n}$, a general smooth hypersurface $H \in |\mathscr{A}^d|$ is Kobayashi hyperbolic.*

The proof of Theorem 0.2 in [3] is also "almost" effective on $d_{\mathrm{Kob},n}$ because of two noetherianity arguments: the first concerns the increasing sequences of Wronskians ideal sheaves; the second concerns a constant arising in Nakamaye's theorem as that of [4], which can be made effective by Theorem 2.2. Our second goal of the present paper is to give an intrinsic interpretation of Brotbek's Wronskians (see § 1.2), and as a byproduct, we can render the above-mentioned first noetherianity argument effective. This in turn provides effective lower bounds for the Kobayashi conjecture in combination with the explicit formula of $d_{\mathrm{Kob},n}$ in [3].

THEOREM B. – *In the same setting as Theorem 0.2, one can take*

$$d_{\mathrm{Kob},n} = n^{2n+3}(n+1).$$

Let us mention that in [3] Brotbek obtained a much stronger result than Theorem 0.2. Indeed, he proved that for the hypersurface H in Theorem 0.2, the tautological line bundle $\mathscr{O}_{H_k}(a_k, \ldots, a_1)$ on the *Demailly-Semple k-jet tower* H_k of the direct manifold (H, T_H) is "almost ample" for some $(a_1, \ldots, a_k) \in \mathbb{N}^k$ when $k \geqslant n-1 = \dim H$. In view of the following vanishing theorem by Diverio in [13], the above-mentioned lower bound for k in [3] is optimal.

THEOREM 0.3 (Diverio). – *Let $Z \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of any degree in $\mathbb{P}^n$. Then*

$$H^0(Z, E^{\mathrm{GG}}_{k,m} T^*_Z) = 0$$

for all $m \geqslant 1$ and $1 \leqslant k < \dim(Z)/\mathrm{codim}(Z)$. Here $E^{\mathrm{GG}}_{k,m} T^*_Z$ denotes the Green-Griffiths jet bundle *of order k and weighted degree m.*

Motivated by the above vanishing theorem, in the same vein as the Debarre conjecture, Diverio-Trapani proposed the following generalized conjecture in [16].

CONJECTURE 0.4 (Diverio-Trapani). – *Let $Z \subset \mathbb{P}^n$ be the complete intersection of c-general hypersurfaces of sufficiently high degree. Then the* invariant jet bundle $E_{k,m}T_Z^*$ *is ample provided that $k \geqslant \frac{n}{c} - 1$ and $m \gg 0$.*

The last aim of the present paper is to study Conjecture 0.4 using geometric methods in [4, 3].

THEOREM C. – *Let X be an n-dimensional projective manifold equipped with a very ample line bundle $\mathscr{A}$, and let $Z \subset X$ be the complete intersection of c-general hypersurfaces $H_1 \ldots, H_c \in |\mathscr{A}^d|$. Then Z is* almost k-jet ample *(see Definition 1.2) if $k \geqslant \frac{n}{c} - 1$, and $d \geqslant 2cn^{c\lceil \frac{n}{c}\rceil+1} \cdot \lceil \frac{n}{c}\rceil^{c\lceil \frac{n}{c}\rceil+3}$. In particular, Z is Kobayashi hyperbolic.*

Let us mention that we apply the results in the first part of the present paper to obtain the effective lower degree bounds in Theorem C.

In view of the correspondence between tautological line bundles on the Demailly-Semple jet towers and invariant jet bundles studied in [9, Proposition 6.16], the following result on Conjecture 0.4 is a consequence of Theorem C.

COROLLARY D. – *In the same setting as Theorem C, for any $k \geqslant \frac{n}{c} - 1$, there exists a subbundle $\mathscr{F} \subset E_{k,m}T_Z^*$ for some $m \gg 0$ such that*

(i) *$\mathscr{F}$ is ample.*
(ii) *For any regular germ of curve $f : (\mathbb{C}, 0) \to (Z, z)$, there is a global section $P \in H^0(Z, \mathscr{F} \otimes \mathscr{A}^{-1})$ so that $P([f]_k)(0) \neq 0$.*

In other words, one can find a subbundle $\mathscr{F}$ of the invariant jet bundle $E_{k,m}T_Z^*$, which is ample, and the *Demailly-Semple locus* (see [15, §2.1] for the definition) induced by $\mathscr{F}$ is empty.

Lastly, let us mention that the techniques in [4, 3] were extended by Brotbek and the author to prove a logarithmic analogue of the Debarre conjecture in [5], and to prove the logarithmic (orbifold) Kobayashi conjecture in [6]. To achieve the effective lower degree bounds, both the articles [5, 6] rely on the methods in the present paper.

This paper is organized as follows. In § 1.1 we recall the fundamental tools of jet differentials by Demailly, Green-Griffiths and Siu, which can be seen as higher order analogues of symmetric differential forms and provide obstructions to the existence of entire curves. § 1.2 is devoted to the study of new techniques of Wronskians introduced by Brotbek in his proof of the Kobayashi conjecture [3]. We bring a new perspective of Brotbek's Wronskians, which we interpret as a certain *morphism of $\mathscr{O}$-modules* from the jet bundles of a line bundle to the invariant jet bundles. In view of this result one can immediately make the first noetherianity argument in [3] effective. In § 2, by means of an explicit construction of global sections with a "negative twist", we obtain a *slightly weaker but effective* Nakamaye's theorem for the universal families of zero-dimensional subschemes introduced in [4, 3]. This in turn renders the second noetherianity argument in [3] as well as that in [4] effective, and in combination with the formulas for lower degree bounds in [4, 3], we prove Theorems A and B. The aim