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QUADRIC RANK LOCI ON MODULI OF CURVES AND $K3$ SURFACES

BY GAVRIL FARKAS AND RICHÁRD RIMÁNYI

ABSTRACT. – Assuming that $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ is a morphism of vector bundles on a variety X, we compute the class of the locus in X where $\mathrm{Ker}(\phi)$ contains a quadric of prescribed rank. Our formulas have many applications to moduli theory: (i) we find a simple proof of Borcherds' result that the Hodge class on the moduli space of polarized $K3$ surfaces of fixed genus is of Noether-Lefschetz type, (ii) we construct an explicit canonical divisor on the Hurwitz space parametrizing degree k covers of $\mathbf{P}^1$ from curves of genus $2k-1$, (iii) we provide a closed formula for the Petri divisor on $\overline{\mathcal{M}}_g$ of canonical curves which lie on a rank 3 quadric and (iv) we construct myriads of effective divisors of small slope on $\overline{\mathcal{M}}_g$.

RÉSUMÉ. – Étant donné deux fibrés vectoriels $\mathcal{E}$ et $\mathcal{F}$ sur une variété X et une application de $\mathrm{Sym}^2(\mathcal{E})$ dans $\mathcal{F}$, nous calculons la classe de cohomologie du lieu en X où le kernel de cette application contient une quadrique de rang donné. Nos formules ont plusieurs applications à la théorie d'espaces des modules: (i) nous trouvons une preuve simple du théorème de Bocherds qui établit que la classe de Hodge dans l'espace de modules de surfaces K3 polarisés avec genre fixé, est du type Noether-Lefschetz, (ii) nous construisons un diviseur canonique explicite dans l'espace d'Hurwitz paramétrisant les applications de degré k de courbes du genre $2k-1$ sur la droite projective, (iii) nous fournissons une formule fermée pour le diviseur de Petri dans l'espace de modules de courbes consistant de courbes canoniques contenues d'une quadrique de rang 3 et (iv) nous construisons une myriade de diviseurs de petite pente dans $\overline{\mathcal{M}}_g$.

1. Introduction

Let X be an algebraic variety and let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles on X having ranks e and f respectively. Assume we are given a morphism of vector bundles

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}.$$

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For a positive integer $r \leq e$, we define the subvariety of X consisting of points for which $\mathrm{Ker}(\phi)$ contains a quadric of corank at least r, that is,

$$\overline{\Sigma}^r_{e,f}(\phi) := \Big\{x \in X : \exists\, 0 \neq q \in \mathrm{Ker}(\phi(x)) \ \text{ with } \ \mathrm{rk}(q) \leq e - r\Big\}.$$

Since the codimension of the variety of symmetric $e\times e$-matrices of corank r is equal to $\binom{r+1}{2}$, it follows that the expected codimension of the locus $\overline{\Sigma}^r_{e,f}(\phi)$ is equal to $\binom{r+1}{2}-\binom{e+1}{2}+f+1$. A main goal of this paper is to explicitly determine the cohomology class of this locus in terms of the Chern classes of $\mathscr{E}$ and $\mathscr{F}$. This is achieved for every e, f and r in Theorem 4.4, using a localized Atiyah-Bott type formula. Of particular importance in moduli theory is the case when this locus is expected to be a divisor, in which case our general formula has a very simple form:

THEOREM 1.1. – *We fix integers* $0 \leq r \leq e$ *and set* $f := \binom{e+1}{2} - \binom{r+1}{2}$. *Suppose* $\phi : \mathrm{Sym}^2(\mathscr{E}) \to \mathscr{F}$ *is a morphism of vector bundles over* X. *The class of the virtual divisor* $\overline{\Sigma}^r_{e,f}(\phi)$ *is given by the formula*

$$[\overline{\Sigma}^r_{e,f}(\phi)] = A^r_e\Big(c_1(\mathscr{F}) - \frac{2f}{e}c_1(\mathscr{E})\Big) \in H^2(X,\mathbb{Q}),$$

where

$$A^r_e := \frac{\binom{e}{r}\binom{e+1}{r-1}\cdots\binom{e+r-1}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{2}\cdots\binom{2r-1}{r-1}}.$$

The quantity A^r_e is the degree of the variety of symmetric $e\times e$-matrices of corank at least r inside the projective space of all symmetric $e \times e$ matrices, see [31].

Before introducing a second type of degeneracy loci, we give a definition. If V is a vector space, a pencil of quadrics $\ell \subseteq \mathbf{P}(\mathrm{Sym}^2(V))$ is said to be *degenerate* if the intersection of ℓ with the discriminant divisor $D(V) \subseteq \mathbf{P}(\mathrm{Sym}^2(V))$ is non-reduced. We consider a morphism $\phi : \mathrm{Sym}^2(\mathscr{E}) \to \mathscr{F}$ such that all kernels are expected to be pencils of quadrics and impose the condition that the pencil be degenerate.

THEOREM 1.2. – *We fix integers* e *and* $f = \binom{e+1}{2} - 2$ *and let* $\phi : \mathrm{Sym}^2(\mathscr{E}) \to \mathscr{F}$ *be a morphism of vector bundles. The class of the virtual divisor* $\mathfrak{Dp} := \{x \in X : \mathrm{Ker}(\phi(x))$ *is a degenerate pencil*$\}$ *equals*

$$[\mathfrak{Dp}] = (e-1)\Big(ec_1(\mathscr{F}) - (e^2+e-4)c_1(\mathscr{E})\Big) \in H^2(X,\mathbb{Q}).$$

Theorems 1.1 and 1.2 are motivated by fundamental questions in moduli theory and in what follows we shall discuss some of these applications, which are treated at length in the paper.

Tautological classes on moduli of polarized $K3$ surfaces. – Let $\mathcal{F}_g$ be the moduli space of quasi-polarized $K3$ surfaces $[X, L]$ of genus g, that is, satisfying $L^2 = 2g - 2$. We denote by $\pi : \mathcal{X} \to \mathcal{F}_g$ the universal $K3$ surface and choose a polarization line bundle $\mathcal{L}$ on $\mathcal{X}$. We consider the Hodge class

$$\lambda := c_1\big(\pi_*(\omega_\pi)\big) \in CH^1(\mathcal{F}_g).$$

Note that $CH^1(\mathcal{F}_g) \cong H^2(\mathcal{F}_g, \mathbb{Q})$. Inspired by Mumford's definition of the κ classes on $\mathcal{M}_g$, for integers $a, b \geq 0$, Marian, Oprea and Pandharipande [39] introduced the classes $\kappa_{a,b} \in CH^{a+2b-2}(\mathcal{F}_g)$ whose definition we recall in Section 9. In codimension 1, there are two such classes, namely

$$\kappa_{3,0} := \pi_*\Big(c_1(\mathcal{L})^3\Big) \text{ and } \kappa_{1,1} := \pi_*\Big(c_1(\mathcal{L}) \cdot c_2(\mathcal{T}_\pi)\Big) \in CH^1(\mathcal{F}_g).$$

Both these classes depend on the choice of $\mathcal{L}$, but the following linear combination

$$\gamma := \kappa_{3,0} - \frac{g-1}{4}\kappa_{1,1} \in CH^1(\mathcal{F}_g)$$

is intrinsic and independent of the polarization line bundle.

For a general element $[X, L] \in \mathcal{F}_g$ one has $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$. Imposing the condition that $\mathrm{Pic}(X)$ be of rank at least 2, one is led to the notion of Noether-Lefschetz (NL) divisor on $\mathcal{F}_g$. For non-negative integers h and d, we denote by $D_{h,d}$ the locus of quasi-polarized $K3$ surfaces $[X, L] \in \mathcal{F}_g$ such that there exists a primitive embedding of a rank 2 lattice

$$\mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot D \subseteq \mathrm{Pic}(X),$$

where $D \in \mathrm{Pic}(X)$ is a class such that $D \cdot L = d$ and $D^2 = 2h - 2$. From the Hodge Index Theorem $D_{h,d}$ is empty unless $d^2 - 4(g-1)(h-1) > 0$. Whenever non-empty, $D_{h,d}$ is pure of codimension 1.

Maulik and Pandharipande [40] conjectured that $\mathrm{Pic}(\mathcal{F}_g)$ is spanned by the Noether-Lefschetz divisors $D_{h,d}$. This has been recently proved in [6] using deep automorphic techniques. Note that the rank of $\mathrm{Pic}(\mathcal{F}_g)$ can become arbitrarily large and understanding all the relations between NL divisors remains a daunting task. Borcherds [8] using automorphic forms on $O(2, n)$ has shown that the Hodge class λ is supported on NL divisors. A second proof of this fact, via Gromov-Witten theory, is due to Pandharipande and Yin, see [46] Section 7. Using Theorem 1.1, we find very simple and explicit Noether-Lefschetz representatives of both classes λ and γ. Our methods are within the realm of algebraic geometry and we use no automorphic forms.

We produce relations among tautological classes on $\mathcal{F}_g$ using the projective geometry of embedded $K3$ surfaces of genus g. We study geometric conditions that single out *only* NL special K3 surfaces. Let us first consider the divisor in $\mathcal{F}_g$ consisting of $K3$ surfaces which lie on a rank 4 quadric. We fix a $K3$ surface $[X, L] \in \mathcal{F}_g$ with $g \geq 4$ and let $\varphi_L : X \to \mathbf{P}^g$ be the morphism induced by the polarization L. One computes $h^0(X, L^{\otimes 2}) = 4g-2$. Assuming that the image $X \subseteq \mathbf{P}^g$ is projectively normal (which holds under very mild genericity assumptions, see again Section 9), we observe that the space $I_{X,L}(2)$ of quadrics containing X has the following dimension:

$$\dim I_{X,L}(2) = \dim \mathrm{Sym}^2 H^0(X, L) - h^0(X, L^{\otimes 2}) = \binom{g-2}{2}.$$