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THE FRANKS-MISIUREWICZ CONJECTURE FOR EXTENSIONS OF IRRATIONAL ROTATIONS

BY ANDRES KOROPECKI, ALEJANDRO PASSEGGI AND MARTÍN SAMBARINO

ABSTRACT. – We show that a toral homeomorphism which is homotopic to the identity and topologically semiconjugate to an irrational rotation of the circle is always a pseudo-rotation (i.e., its rotation set is a single point). In combination with recent results, this allows us to complete the study of the Franks-Misiurewicz conjecture in the minimal case.

RÉSUMÉ. – On montre qu'un homéomorphisme du tore homotope à l'identité et topologiquement semiconjugué à une rotation irrationnelle du cercle est une pseudo-rotation (c'est-à-dire, son ensemble de rotation se réduit à un point). À l'aide de résultats récents, ceci conclut l'étude de la conjecture de Franks-Misiurewicz pour les homéomorphismes minimaux.

1. Introduction

It is a general goal in mathematics to classify objects by means of simpler invariants associated to them. In the study of the dynamics of surface maps, the rotation set is a prototypical example of this approach. Being a natural generalization in different contexts of the Poincaré rotation number of orientation preserving circle homeomorphisms, it provides basic dynamical information for surface maps in the homotopy class of the identity [20, 23, 9].

In the two dimensional torus, it can be said that a theory has emerged supported on this invariant (see [20] for a wide exposition). If F is a lift of a torus homeomorphism f in the homotopy class of the identity, its rotation set is defined by

$$\rho(F) = \left\{ \lim_{i\to\infty} \frac{F^{n_i}(x_i) - x_i}{n_i} \; : \; \text{where } n_i \nearrow +\infty, \; x_i \in \mathbb{R}^2 \right\}.$$

In the seminal article [20] Misiurewicz and Ziemian proved the convexity and compactness of rotation sets. The finite nature for the possible geometries of a convex set in the plane given by points, non-trivial line segments, or convex sets with nonempty interior, allowed to start a systematic study based on these three cases.

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Results concerning this theory can be classified in two different directions. A first direction aims to obtain interesting dynamical information from knowing the geometry of the rotation set, where the list of results is huge. For instance it is known that when the rotation set has nonempty interior the map has positive topological entropy [19] and abundance of periodic orbits and ergodic measures [8, 7, 21]; bounded deviations properties are found both for the nonempty interior case and the non-trivial segment case [6, 1, 18, 15] (see also [2, 22] as possible surveys (1)).

A second direction aims to establish which kind of convex sets can be realized as rotation sets. Here we find fundamental problems which remain unanswered (compared with the first direction, it can be said, the state of art is considerably underdeveloped). For convex sets having nonempty interior, all known examples achieved as rotation sets have countably many extremal points [16, 5]. For rotation sets with empty interior, there is a long-standing conjecture due to Franks and Misiurewicz [10], which is the matter of this work. The conjecture aims to classify the possible rotation sets with nonempty interior, and it states that any such rotation set is either a singleton or a non-trivial line segment I which falls in one of the following cases:

(i) I has rational slope and contains rational points (2);

(ii) I has irrational slope and one of the endpoints is rational.

For case (ii) A. Avila has presented a counterexample in 2014 (3), where a non-trivial segment with irrational slope containing no rational points is obtained as rotation set. Moreover, the counterexample is minimal ($\mathbb{T}^2$ is the unique compact invariant set) and C^∞, among other interesting features.

Still concerning case (ii), P. Le Calvez and F. A. Tal showed that whenever the rotation set is a non-trivial segment with irrational slope and containing a rational point, this rational point must be an endpoint of the segment [18], so segments of irrational slope containing rational points obey the conjecture.

Item (i), however, remains open: is it true that the only nontrivial segments of rational slope realized as a rotation set are those containing rational points? Although partial progress has been made in recent years [11, 14, 13, 15], the question remained open even in the minimal case.

In this article we prove that, in contrast to Avila's counter example, case (i) in the conjecture is true for minimal homeomorphisms. As we see in the next paragraph, we prove that case (i) must hold in the family of *extensions of irrational rotations* which in particular provides the answer for minimal homeomorphisms.

1.1. Precise statement, context and scope.

The family of *extensions of irrational rotations* is given by those toral homeomorphisms in the homotopy class of the identity which are topologically semi-conjugate to an irrational rotation of the circle. The study of the conjecture in this particular family was introduced in [13], following a program by T. Jäger: supported in the ideas presented in [11], one may first

(1) Unfortunately both surveys are far from being up to date.
(2) I.e., points with both coordinates rational.
(3) Still unpublished.

aim to show that every possible counter example for the rational case (i) in the conjecture must be contained in this family, and as a second step one may study the conjecture in the class of extensions of irrational rotations. There is significant progress in the first step of the program under some recurrence assumptions [11, 14, 15]. On the other hand, for the second step the only known result states that if a counter-example exists, the fibers of the conjugation must be topologically complicated [13]. This sole fact does not lead to a contradiction, since such a fiber structure is possible for extensions of irrational rotations (see [3]). Our main result in this article completely solves the second step of Jäger's program: there are no counter examples to the Franks-Misiurewicz conjecture in the family of extensions of irrational rotations.

The rotation set of an extension of an irrational rotation in $\mathbb{T}^2$ contains no rational points, and it must be either a singleton or an interval of rational slope (see for instance [13]). In [14] it is proved that every area-preserving homeomorphism homotopic to the identity having a *bounded deviations* property is an extension of an irrational rotation (see also [11]). Recently A. Kocsard showed that minimal homeomorphisms having a non-trivial interval with rational slope as rotation set have the *bounded deviations* property [15], and as a consequence every minimal homeomorphism having a non-trivial interval with rational slope as rotation set must be an extension of an irrational rotation.

Our main result is the following:

THEOREM 1. – *The rotation set of a lift of any extension of an irrational rotation is a singleton.*

Using the previously mentioned results we find that case (i) in the Franks-Misiurewicz conjecture is true for minimal homeomorphisms:

THEOREM 2. – *The rotation set of a lift of any minimal homeomorphism of* $\mathbb{T}^2$ *homotopic to the identity is either:*

(i) *a single point of irrational coordinates, or*

(ii) *a segment with irrational slope containing no rational points.*

Note that both cases are realized; the first one by minimal rotations, and the second by Avila's example.

In the case of diffeomorphisms, J. Kwapisz has shown that the possible existence of an example whose rotation set is an interval contained in a line of irrational slope having a rational point outside the interval is equivalent to the existence of an example with a non-trivial segment of rational slope containing no rational points [17]. This was adapted to the C^0 setting by Béguin, Crovisier and Le Roux [4]. Using these results, we have the following:

COROLLARY. – *Case* (ii) *of the previous theorem can only hold if the supporting line of the segment contains no rational points.*