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EXISTENCE OF EXPANDERS OF THE HARMONIC MAP FLOW

BY ALIX DERUELLE AND TOBIAS LAMM

ABSTRACT. – We investigate the existence of weak expanding solutions of the harmonic map flow for maps with values into a smooth closed Riemannian manifold. We prove the existence of such solutions in the case the target manifold is isometrically embedded as a hypersurface of some Euclidean space and the initial condition is a Lipschitz map that is homotopic to a constant. Regularity is proved outside a compact set.

RÉSUMÉ. – Nous nous intéressons à l'existence de solutions faibles au flot d'applications harmoniques pour des applications à valeurs dans une variété riemannienne compacte lisse et sans bord. Nous montrons l'existence de telles solutions dans le cas où la variété cible peut-être plongée isométriquement comme une hypersurface de l'espace euclidien et si la condition initiale est une application lipschitzienne homotope à une constante. La question de la régularité est également traitée.

1. Introduction

In this paper we consider the Cauchy problem for the heat flow of harmonic maps $(u(t))_{t\geq 0}$ from $\mathbb{R}^n$, $n \geq 3$ to a closed smooth Riemannian manifold (N^{m-1}, g) isometrically embedded as a hypersurface in some Euclidean space $\mathbb{R}^m$, $m \geq 2$. More precisely, we study the parabolic system

$$(1)\qquad \begin{cases} \partial_t u = \Delta u + A(u)(\nabla u, \nabla u), & \text{on } \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} = u_0, \end{cases}$$

for a given map $u_0 : \mathbb{R}^n \to N$, where $A(u)(\cdot,\cdot) : T_uN \times T_uN \to (T_uN)^{\perp}$ denotes the second fundamental form of the embedding $N^{m-1} \hookrightarrow \mathbb{R}^m$ evaluated at u. Note that the equation (1) is equivalent to $\partial_t u - \Delta u \perp T_uN$ for a family of maps $(u(t))_{t\geq 0}$ which map into N. Recall that this evolution equation is invariant under the scaling

$$(2)\qquad (u_0)_\lambda(x) := u_0(\lambda x), \quad x \in \mathbb{R}^n,$$

$$(3)\qquad u_\lambda(x,t) := u(\lambda x, \lambda^2 t), \quad \lambda > 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+.$$

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If u_0 is invariant under the above scaling, i.e., if u_0 is 0-homogeneous, solutions of the harmonic map flow which are invariant under scaling are potentially well-suited for smoothing out u_0 instantaneously. Such solutions are called expanding solutions or expanders. In this setting, it turns out that (1) is equivalent to a static equation, i.e., that it does not depend on time anymore. Indeed, if u is an expanding solution in the previous sense then the map $U(x) := u(x,1)$ for $x \in \mathbb{R}^n$, satisfies the elliptic system

$$\begin{cases} \Delta_f U + A(U)(\nabla U, \nabla U) = 0, & \text{on } \mathbb{R}^n, \\ \lim_{r\to+\infty} U(r,\omega) = u_0(\omega), & (r,\omega) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}, \end{cases} \tag{4}$$

where f and Δ_f are defined by

$$f(x) := \frac{|x|^2}{4} + \frac{n}{2}, \quad x \in \mathbb{R}^n,$$
$$\Delta_f U := \Delta U + \nabla f \cdot \nabla U = \Delta U + \frac{r}{2}\partial_r U.$$

The function f is called the potential function and it is defined up to an additive constant. The choice of this constant is dictated by the requirement

$$\Delta_f f = f.$$

The operator Δ_f is called a weighted laplacian and it is unitarily conjugate to a harmonic oscillator $\Delta - |x|^2/16$.

Conversely, if U is a solution to (4) then the map $u(x,t) := U(x/\sqrt{t})$, for $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, is a solution to (1). Because of this equivalence, u_0 can be interpreted either as an initial condition or as a boundary data at infinity.

The interest in expanding solutions is basically due to two reasons. On the one hand, these scale invariant solutions are important with respect to the continuation of a weak harmonic map flow between two closed Riemannian manifolds. Indeed, by the work of Chen and Struwe [3], there always exists a weak solution of the harmonic map flow starting from a smooth map between two closed Riemannian manifolds. It turns out that such a flow is not always smooth and the appearance of singularities is caused by either non-constant 0-homogeneous harmonic maps defined from $\mathbb{R}^n$ to N, the so called tangent maps, or shrinking solutions (also called quasi-harmonic spheres) that are ancient solutions invariant under scaling. Expanding solutions can create an ambiguity in the continuation of the flow after it reaches a singularity by a gluing process. On the other hand, one might be interested in using the smoothing effect of the harmonic map flow. More precisely, it is tempting to attach a canonical map to any map between (stratified) manifolds with prescribed singularities. It turns out that 0-homogeneous maps are the building blocks of such singularities and expanding solutions are likely to be the best candidates to do this job.

In this paper, we investigate the question of existence of expanding solutions coming out of u_0 in the case where there is no topological obstruction, i.e., under the assumption that u_0 is homotopic to a constant when restricted to $\mathbb{S}^{n-1}$. Our main result is the following

THEOREM 1.1. – *Let $n \geq 3$ and $m \geq 2$ be two integers and let $u_0 : \mathbb{R}^n \to (N^{m-1}, g) \subset \mathbb{R}^m$ be a Lipschitz 0-homogeneous map such that its restriction to $\mathbb{S}^{n-1}$ is homotopic to a constant.*

Then there exists a weak expanding solution $u(\cdot, 1) =: U(\cdot)$ *of the harmonic map flow coming out of* u_0 *weakly which is regular off a closed singular set with at most finite* $(n-2)$*-dimensional Hausdorff measure. Moreover, there exist a radius* $R = R(\|\nabla u_0\|_{L^2_{\mathrm{loc}}}, n, m) > 0$ *and a constant* $C = C(\|\nabla u_0\|_{L^2_{\mathrm{loc}}}, n, m) > 0$ *such that* U *is smooth outside* $B(0, R)$ *and,*

$$
\begin{aligned}
|\nabla U|(x) &\leq \frac{C}{|x|}, \quad |x| \geq R, \\
\|\nabla u(t)\|_{L^2(B(x_0,1))} &\leq C\left(n, m, \|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, t\right) \|\nabla u_0\|_{L^2(B(x_0,1))}, \quad \forall x_0 \in \mathbb{R}^n, \\
\|\partial_t u\|_{L^2((0,t),L^2_{\mathrm{loc}}(\mathbb{R}^n))} &\leq C(n, m, t)\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)},
\end{aligned}
$$

where $\lim_{t\to 0} C\left(n, m, \|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, t\right) = \lim_{t\to 0} C(n, m, t) = 1$.

In particular, $u(\cdot, t)$ *tends to* u_0 *as* t *goes to* 0 *in the* $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ *sense and if* u_0 *is not harmonic then* $u(\cdot, t)$ *is not constant in time. Finally, one has the following convergence rate:*

$$
|U(x) - u_0(x/|x|)| \leq C|x|^{-1}, \quad |x| \geq R. \tag{5}
$$

We remark that the regularity result of the theorem is reminiscent of and based on the fundamental work of Chen and Struwe [3] and Cheng [4]. We localize their approach to ensure the smoothness of the solution outside a closed ball since the local energy is decaying to 0 at infinity. This lets us establish a sharp convergence rate for Lipschitz maps. Theorem 1.1 and its proof provide the existence of a non constant in time (or equivalently non radial) expanding solution in case the initial map is not harmonic. Since the initial condition u_0 is allowed to have large local-in-space energy, it is likely that uniqueness will fail. In particular, the authors do not know if the solution produced by Theorem 1.1 coming out of a 0-homogeneous harmonic map will stay harmonic.

Let us make some comments about the proof of Theorem 1.1 before we describe its main steps. A direct perturbative approach is well-suited in case the target Riemannian manifold (N, g) has non-positive sectional curvature as shown by the second author [7] or if the $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ energy of u_0 is assumed to be arbitrarily small: see Section 2 for a proof. One more instance where such a direct approach works well is by imposing further symmetry on the initial condition u_0 and the target manifold N as initiated by Germain and Rupflin [9]. To conclude, the nonlinearity of the target manifold and the potential formation of finite time singularities are the two main obstacles to a direct perturbative approach in general.

To circumvent this issue, we follow Chen-Struwe's penalisation procedure [3] and we construct our expanding solution to the harmonic map flow as a limit of expanding solutions starting from the same initial condition u_0 of a so called homogeneous Chen-Struwe flow with parameter K, see Section 3.3 for more definitions. Before stating the main result about this flow, we recall some definitions.

Let $n \geq 3$ and let $u_0 : \mathbb{R}^n \to (N, g) \subset \mathbb{R}^m$ be a 0-homogeneous map. Let us notice that (N, g) is not assumed to be a hypersurface of $\mathbb{R}^m$ at this stage.