ISSN 0012-9593

quatrième série - tome 54 fascicule 5 septembre-octobre 2021

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Julien MARCHÉ & Ramanujan SANTHAROUBANE

Asymptotics of quantum representations of surface groups

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 437 euros.
Abonnement avec supplément papier :
Europe : 600 €. Hors Europe : 686 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 54, 2021, p. 1275 à 1296

ASYMPTOTICS OF QUANTUM REPRESENTATIONS OF SURFACE GROUPS

BY Julien MARCHÉ AND Ramanujan SANTHAROUBANE

ABSTRACT. – For a banded link L in a surface times a circle, the Witten-Reshetikhin-Turaev invariants are topological invariants depending on a sequence of complex $2p$-th roots of unity $(A_p)_{p\in 2\mathbb{N}}$. We show that there exists a polynomial P_L such that these normalized invariants converge to $P_L(u)$ when A_p converges to u, for all but a finite number of u's in S^1. This is related to the AMU conjecture which predicts that non-simple curves have infinite order under quantum representations (for big enough levels). Estimating the degree of P_L, we exhibit particular types of curves which satisfy this conjecture. Along the way we prove the Witten asymptotic conjecture for links in a surface times a circle.

RÉSUMÉ. – Pour un entrelacs en bande L dans le produit d'une surface par un cercle, les invariants de Witten-Reshetikhin-Turaev sont des invariants topologiques dépendant d'une suite de racines $2p$-ièmes de l'unité $(A_p)_{p\in 2\mathbb{N}}$. Nous montrons qu'il existe un polynôme P_L tel que ces invariants normalisés convergent vers $P_L(u)$ quand A_p tend vers u, sauf pour un nombre fini de u dans S^1. Ceci est relié à la conjecture AMU qui prédit que les courbes non simples sont d'ordre infini dans la représentation quantique (en niveau assez grand). En estimant le degré de P_L, on exhibe certaines courbes qui satisfont la conjecture. En chemin, nous prouvons la conjecture asymptotique de Witten pour les entrelacs dans le produit d'une surface par un cercle.

1. Statement of the results

1.1. Motivation and Main result

This paper is concerned with invariants arising from Witten-Reshetikhin-Turaev SU(2) topological quantum field theories (TQFT) following the skein theoretical approach of [3]. Such a TQFT defines for M an oriented compact 3-manifold without boundary and $L \subset M$ a banded link, a sequence of invariants $Z_p(M, L)$ indexed by even integers $p = 2r \geq 6$. For a given p, the invariant $Z_p(M, L)$ belongs to the cyclotomic field [(1)] $K_p = \mathbb{Q}[A]/(\phi_{2p}(A))$,where ϕ_{2p} denotes the cyclotomic polynomial, that is the (monic)

(1) Indeed in a finite extension of it, but we will not need it here.

 doi:10.24033/asens.2481

minimal polynomial over $\mathbb{Q}$ of $e^{i\pi/p}$. To have a numerical invariant, one needs to specify an embedding of K_p into $\mathbb{C}$ or equivalently a $2p$-th primitive root of unity $A_p \in \mathbb{C}$. We will denote by $\mathrm{ev}_{A_p} Z_p(M,L) \in \mathbb{C}$ the associated numerical evaluation. An interesting question is to understand the asymptotic of the quantity $\mathrm{ev}_{A_p} Z_p(M,L)$ as $p \to \infty$ and as A_p converges to a given number on the unit circle. When $A_p = -e^{i\pi/p}$, this problem is called the Witten asymptotic expansion conjecture. Other limits have not been studied yet with the exception of some Seifert spaces studied by Lawrence and Zagier, see [12].

In this paper, we focus on the case $M = \Sigma \times S^1$ where Σ is a compact connected oriented closed surface. We look for a formula for the quantum invariant

$$\mathrm{tr}_p(L) = \frac{Z_p(\Sigma \times S^1, L)}{Z_p(\Sigma \times S^1, \emptyset)} \in K_p,$$

where $L \subset \Sigma \times S^1$ is a given banded link. Notice that the quantity $Z_p(\Sigma \times S^1, \emptyset)$ is the dimension of $V_p(\Sigma)$: the K_p-vector space associated to Σ by the Witten-Reshetikhin-Turaev TQFT. Moreover, $\dim V_p(\Sigma)$ is computed by the Verlinde formula and is polynomial in p with degree $0, 1, 3g-3$ if the genus of Σ is $g = 0, 1$ or $g \geq 2$ respectively. Hence the asymptotics of the quantity $\mathrm{ev}_{A_p} Z_p(\Sigma \times S^1, L)$ is determined by the asymptotics of $\mathrm{ev}_{A_p} \mathrm{tr}_p(L)$. The main result of this paper is that the asymptotics of $\mathrm{ev}_{A_p} \mathrm{tr}_p(L)$ is almost determined by the evaluation of a Laurent polynomial with integral coefficients depending only on L.

THEOREM 1.1. – *Let L be a link in $\Sigma \times S^1$. There exist a Laurent polynomial $P_L \in \mathbb{Z}[A^{\pm 1}]$ and a finite set $\Omega_L \subset S^1$ such that for any sequence $\{A_p\}_{p \in 2\mathbb{N}}$ such that $A_p \underset{p\to\infty}{\longrightarrow} u \notin \Omega_L$, one has*

$$\mathrm{ev}_{A_p} \mathrm{tr}_p(L) = P_L(u) + O\Big(\frac{1}{p}\Big).$$

In particular the polynomial $P_L \in \mathbb{Z}[A^{\pm 1}]$ is well-defined and is a topological invariant of L.

The polynomial P_L can be viewed as a generalization of the Kauffman bracket of a link in S^3. Indeed if $L \subset B^3$ is a link inside a 3-ball embedded in $\Sigma \times S^1$, the polynomial P_L is nothing but the usual Kauffman bracket of L. For a more complicated link inside $\Sigma \times S^1$, this polynomial can be computed algorithmically, see Section 2. The existence of such a polynomial is not clear a priori, we note that Gilmer already built one in the case of a connected sum of $S^2 \times S^1$'s using other methods (see [7]). A similar comment can be made on Costantino's work (see [5]).

1.2. Cyclic expansions

In order to prove Theorem 1.1, we introduce the key notion of *cyclic expansion*. We will denote by $\mathcal{C}$ the vector space of maps $f : \mathbb{R} \to \mathbb{R}$ which are piecewise polynomial with compact support.

DEFINITION 1.2. – We will say that the sequence $\mathrm{tr}_p(L)$ has a cyclic expansion if there exist $P_L \in \mathbb{Z}[A^{\pm 1}]$, an integer $\beta \geq 0$ and a family $f_0, \ldots, f_{2\beta-1} \in \mathcal{C}$ such that

$$\mathrm{tr}_p(L) = P_L(A) + \frac{1}{p} \sum_{\alpha=0}^{2\beta-1} \sum_{n \in \mathbb{Z}} A^{2\beta n + \alpha} f_\alpha\left(\frac{n}{p}\right) + O\left(\frac{1}{p}\right). \tag{1}$$

We need to explain the meaning of $O(\frac{1}{p})$ in Equation (1). Indeed, we have to interpret both sides as elements of $K_p\otimes\mathbb{R}$ endowed with the norm $||x||_p = \inf\{\max_n |c_n|, x = \sum_{n=0}^{p-1} c_n A^n\}$.

The main technical result of this article is the following theorem.

THEOREM 1.3. – *For any banded link $L \subset \Sigma \times S^1$, the sequence $\mathrm{tr}_p(L)$ admits a cyclic expansion.*

The proof of this theorem consists in a careful counting of integral points in various polytopes related to TQFT. It is postponed to Section 3. The interest of having such a cyclic expansion is that the study of asymptotics is reduced to the following question.

Let f be in $\mathcal{C}$, β be a non zero integer and A_p be a convergent sequence of $2p$-th primitive roots of unity. What is the asymptotics of $\frac{1}{p}\sum_{n\in\mathbb{Z}} A_p^{2\beta n} f(\frac{n}{p})$ as p tends to infinity? This problem can be solved with elementary analytic tools as follows.

PROPOSITION 1.4. – *Let $f \in \mathcal{C}$ and β be a positive integer. Let A_k be a sequence of $2p_k$-th primitive roots of unity with p_k a strictly increasing sequence of even integers.*

1. *If $\lim\limits_{k\to\infty} A_k = u$ with $u^{2\beta} \neq 1$ one has*

$$\frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f\Big(\frac{n}{p_k}\Big) = O\Big(\frac{1}{p_k}\Big).$$

2. *If $\lim\limits_{k\to\infty} A_k = u$ with $u^{2\beta} = 1$ we write $A_k = ue^{i\pi\theta_k}$ so that $\lim\limits_{k\to\infty}\theta_k = 0$. If $p_k\theta_k$ diverges when k goes to infinity then*

$$\frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f\Big(\frac{n}{p_k}\Big) = O\Big(\frac{1}{p_k\theta_k}\Big).$$

3. *In the same setting as (2) suppose that $p_k\theta_k$ does not diverge. As $A_k^{2p_k} = 1$, the sequence $\theta_k p_k$ takes discrete values and up to extracting a subsequence, one can suppose that it is constant equal to $\frac{\sigma}{\beta}$, i. e. $A_k = ue^{\frac{i\pi\sigma}{\beta p_k}}$ for some odd integer σ. Then, we have*

$$\frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f\Big(\frac{n}{p_k}\Big) = \int_{\mathbb{R}} e^{2i\pi x\sigma} f(x)dx + O\Big(\frac{1}{p_k}\Big).$$

Proof. – We set $H_k = \frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f(\frac{n}{p_k})$. We compute

$$(1-A_k^{2\beta})H_k = \frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n}\Big(f\Big(\frac{n}{p_k}\Big) - f\Big(\frac{n-1}{p_k}\Big)\Big) = O\Big(\frac{1}{p_k}\Big).$$

The last equality is obtained by applying a Taylor expansion of f away from a finite number of values: this is possible since f is piecewise polynomial.

In the first case, since $u^{2\beta} \neq 1$, we deduce that $H_k = O(\frac{1}{p_k})$.

In the second case, we simply observe that $1 - A_k^{2\beta} \sim -2i\pi\beta\theta_k$ hence $H_k = O\big(\frac{1}{p_k\theta_k}\big)$ and we can conclude.

In the last case, we write $H_k = \frac{1}{p_k}\sum_{n\in\mathbb{Z}} e^{2i\pi\sigma\frac{n}{p_k}} f(\frac{n}{p_k})$ and recognize a Riemann sum. This gives $H_k = \int_{\mathbb{R}} e^{2i\pi\sigma x} f(x)dx + O\big(\frac{1}{p_k}\big)$. □

We observe that Proposition 1.4 and Theorem 1.3 imply directly Theorem 1.1.