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FLAG-APPROXIMABILITY OF CONVEX BODIES AND VOLUME GROWTH OF HILBERT GEOMETRIES

BY CONSTANTIN VERNICOS AND CORMAC WALSH

ABSTRACT. – We introduce the flag-approximability of a convex body to measure how easy it is to approximate by polytopes. We show that the flag-approximability is exactly half the volume entropy of the Hilbert geometry on the body, and that both quantities are maximized when the convex body is a Euclidean ball.

We also compute explicitly the asymptotic volume of a convex polytope, which allows us to prove that simplices have the least asymptotic volume.

RÉSUMÉ. – Nous introduisons l'approximabilité-drapeau d'un corps convexe qui mesure la difficulté de l'approcher par des polytopes convexes. Nous montrons que l'approximabilité-drapeau d'un corps convexe est égale à la moitié de l'entropie volumique de sa géométrie de Hilbert associée et que les deux invariants sont maximaux lorsque le corps convexe est une boule euclidienne.

Nous calculons également le volume asymptotique de la géométrie de Hilbert d'un polytope convexe, ce qui nous permet de démontrer que les simplexes ont le volume asymptotique minimal.

Introduction

An important problem with many practical applications is to approximate convex bodies with polytopes that are as simple as possible, in some sense. Various measures of complexity of a polytope have been considered in the literature. These include counting the number of vertices, the number of facets, or even the number of faces [3]. One could also use, however, the number of *maximal flags*. Recall that a maximal flag of a d-dimensional polytope is a finite sequence $(f_0, \ldots, f_d)$ of faces of the polytope such that each face f_i has dimension i and is contained in the boundary of f_{i+1}.

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Suppose we wish to approximate a convex body Ω by a polytope within a Hausdorff distance $\varepsilon > 0$. Let $N_{\mathrm{f}}(\varepsilon, \Omega)$ be the least number of maximal flags over all polytopes satisfying this criterion. We define the *flag approximability* of Ω to be

$$a_{\mathrm{f}}(\Omega) := \liminf_{\varepsilon \to 0} \frac{\log N_{\mathrm{f}}(\varepsilon, \Omega)}{-\log \varepsilon}.$$

This is analogous to how Schneider and Wieacker [10] defined the (vertex) approximability, where the least number of vertices was used instead of the least number of maximal flags.

Facet and face approximabilities can also be defined in a similar fashion. It is not known if any equalities hold between the vertex, facet, face, and flag approximabilities. An advantage of using the flag approximability is that one can relate it to the *volume entropy* of the Hilbert metric on the body in the following way.

Choose a base point p in the interior of the convex body Ω, and for each $R > 0$ denote by $B_\Omega(p, R)$ the closed ball centered at p of radius R in the Hilbert geometry. Let Vol^H denote the Holmes-Thompson volume. The (lower) volume entropy of the Hilbert geometry on Ω is defined to be

$$\mathrm{Ent}(\Omega) := \liminf_{R \to \infty} \frac{\log \mathrm{Vol}^H\big(B_\Omega(p, R)\big)}{R}.$$

Observe that this does not depend on the base point p, and moreover does not change if one takes instead the Busemann volume. One can also define the upper flag approximability and the upper volume entropy by taking supremum limits instead of infimum ones. Although the two entropies do not generally coincide, as shown by the first author in [13], all our results and proofs hold when replacing lim inf with lim sup.

Theorem 1. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body. Then,*

$$\mathrm{Ent}(\Omega) = 2a_{\mathrm{f}}(\Omega).$$

The same result concerning the vertex approximability was proved by the first author [13] in dimensions two and three. In higher dimension, it was shown only that the volume entropy is greater than or equal to twice the vertex approximability. The motivation was to try to prove the entropy upper bound conjecture, which states that the volume entropy of any convex body is no greater than $d - 1$. This would follow from equality of the two quantities just mentioned using the well-known result, proved by Fejes-Toth [7] in dimension two and by Bronshteyn-Ivanov [5] in the general case, that the vertex approximability of any convex body is no greater than $(d - 1)/2$.

We show, using a slight modification of the technique in Arya-da Fonseca-Mount [3], that the Bronshteyn-Ivanov bound also holds for the flag approximability.

Theorem 2. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body. Then,*

$$a_{\mathrm{f}}(\Omega) \leq \frac{d-1}{2}.$$

This allows us to deduce the entropy upper bound conjecture. N. Tholozan has also proved this conjecture recently using a different method [11].

COROLLARY 3. – *Let* $\Omega \subset \mathbb{R}^d$ *be a convex body. Then,*

$$\mathrm{Ent}(\Omega) \leq d - 1.$$

For many Hilbert geometries, such as hyperbolic space, the volume of balls grows exponentially. However, for some Hilbert geometries, the volume grows only polynomially. In this case it is useful to make the following definition. Fix some notion of volume Vol. The *asymptotic volume* of the Hilbert geometry on a d-dimensional convex body Ω is defined to be

$$\mathrm{Asvol}(\Omega) := \liminf_{R\to\infty} \frac{\mathrm{Vol}(B_\Omega(p, R))}{R^d}.$$

Note that, unlike in the case of the volume entropy, the asymptotic volume depends on the choice of volume. The first author has shown in [12] that the asymptotic volume of a convex body is finite if and only if the body is a polytope.

In the next theorem, we again see a connection appearing between volume in Hilbert geometries and the number of maximal flags. We denote by $\mathrm{Flags}(\mathscr{P})$ the set of maximal flags of a polytope $\mathscr{P}$. Let Σ be a simplex of dimension d. Observe that $\mathrm{Flags}(\Sigma)$ consists of $(d + 1)!$ elements.

THEOREM 4. – *Let* $\mathscr{P}$ *be a polytope of dimension* d*, and fix some notion of volume* Vol. *Then,*

$$\mathrm{Asvol}(\mathscr{P}) = \frac{|\mathrm{Flags}(\mathscr{P})|}{(d+1)!} \mathrm{Asvol}(\Sigma).$$

An immediate consequence is that the simplex has the smallest asymptotic volume among all convex bodies. This was conjectured by the first author in [12].

COROLLARY 5. – *Let* $\Omega \subset \mathbb{R}^d$ *be a convex body. Then,*

$$\mathrm{Asvol}(\Omega) \geq \mathrm{Asvol}(\Sigma),$$

with equality if and only if Ω *is a simplex.*

Another corollary is the following result, proved originally by Foertsch and Karlsson [8].

COROLLARY 6. – *If the Hilbert geometry on a convex body* Ω *is isometric to a finite-dimensional normed space, then* Ω *is a simplex.*

1. Preliminaries

A *proper* open set in $\mathbb{R}^d$ is an open set not containing a whole line. A non-empty proper open convex set will be called a *convex domain*. The closure of a bounded convex domain is called a *convex body*.