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GLOBAL WELL-POSEDNESS FOR THE 2D STABLE MUSKAT PROBLEM IN $H^{3/2}$

BY DIEGO CÓRDOBA AND OMAR LAZAR

ABSTRACT. – We prove a global existence result of a unique strong solution in $H^{5/2}$ with small $\dot{H}^{3/2}$ semi-norm for the 2D Muskat problem. Hence, allowing the interface to have arbitrary large finite slopes and finite energy (thanks to the L^2 maximum principle). The proof is based on the use of a new formulation of the Muskat equation that involves oscillatory terms. Then, a careful use of interpolation inequalities in homogeneneous Besov spaces allows us to close the a priori estimates.

RÉSUMÉ. – Nous prouvons un résultat d'existence globale d'une unique solution forte dans $H^{5/2}$ en supposant que la semi-norme $\dot{H}^{3/2}$ soit petite pour le problème de Muskat 2D. Ceci permet donc d'avoir des interfaces dont la pente peut être arbitrairement grande et finie (grâce au principe du maximum L^2). La preuve est basée sur l'introduction d'une nouvelle formulation de l'équation de Muskat en termes d'intégrales oscillantes. Ensuite, une utilisation minutieuse d'inégalités d'interpolations dans des espaces de Besov homogènes permet de clore les estimations *a priori*.

1. Introduction

In this paper, we are interested in the Muskat problem which was introduced in [27] by Morris Muskat in order to describe the dynamics of water and oil in sand. The Muskat problem models the motion of an interface separating two incompressible fluids in a porous medium. One can imagine the plane $\mathbb{R}^2$ split into two regions, say $\Gamma_1(t)$, $\Gamma_2(t)$ that evolve with time. We assume that the first region $\Gamma_1(t)$ is occupied by an incompressible fluid with density ρ_1 and the second region $\Gamma_2(t)$ is occupied by another fluid with density ρ_2. We further assume that both fluids are immiscible. The non mixture condition allows one to consider the interface between these two fluids. This interface corresponds to their common boundary $\partial\Gamma_1(t)$ and $\partial\Gamma_2(t)$. The velocity in each region $\Gamma_i(t)$ ($i = 1$ or 2) is governed by the so-called Darcy's law [18], which states that the velocity depends on the gradient pressure,

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the gravity and the density of the fluid (which is transported by the flow) via the following relation,

$$\frac{\mu}{\kappa}u(x,t) = -\nabla p - (0, g\rho), \tag{1.1}$$

where μ is the constant viscosity, κ is the permeability of the porous media and g is the gravity. For the sake of simplicity, we may, without loss of generality, assume that all those constants are equal to 1. The system is then driven by the following transport equation

$$\partial_t \rho + u \cdot \nabla \rho = 0. \tag{1.2}$$

Since the fluids are incompressible we also have

$$\nabla \cdot u = 0. \tag{1.3}$$

Equations (1.1), (1.2) and (1.3) give rise to the so-called incompressible porous media system (IPM). Saffman and Taylor [31] pointed out that in 2D the Muskat problem is similar to the evolution of an interface in a vertical Hele-Shaw cell.

For the Muskat problem we can rewrite the IPM system in terms of the dynamics of the interface in between both fluids (see [1] and [14]). If we denote the interface by a planar curve $z(\alpha,t)$ and if we neglect surface tension, then the interface satisfies

$$\partial_t z(\alpha,t) = \frac{\rho_2 - \rho_1}{2\pi} \int \frac{z_1(\alpha,t) - z_1(\beta,t)}{|z(\alpha,t) - z(\beta,t)|^2} (\partial_\alpha z(\alpha,t) - \partial_\beta z(\beta,t)) d\beta,$$

where the curve z is asymptotically flat at infinity *i.e.,* $(z(\alpha,t) - (\alpha,0)) \to 0$ as $|\alpha| \to \infty$. The point $(0,\infty)$ belongs to $\Gamma_1(t)$, whereas the point $(0,-\infty)$ belongs to $\Gamma_2(t)$. From elementary potential theory, we can derive explicit formulas for the velocity field u and the pressure p from the curve z.

A convenient way of studying the evolution of the interface is to consider this latter as a parametrized graph of a function. When the interface is a graph of a function $z(x,t) = (x, f(x,t))$, then this characterization is preserved locally in time by the system and f satisfies the contour equation

$$f_t(x,t) = -\rho(\Lambda f + T(f)), \tag{1.4}$$

where ρ is equal to $\frac{\rho_2-\rho_1}{2}$, and the operator Λ^γ, $0 < \gamma < 2$, denotes the usual fractional Laplacian operator of order γ and is defined as

$$\Lambda^\gamma f = (-\Delta)^{\gamma/2} f = C_\gamma P.V. \int_{\mathbb{R}} \frac{f(x) - f(x-y)}{|y|^{1+\gamma}} dy,$$

where $C_\gamma > 0$ is a positive constant. In particular, when $\gamma = 1$, the constant is equal to $\frac{1}{\pi}$.

The operator $\mathscr{H}$ denotes the Hilbert transform operator which is defined by

$$\mathscr{H} f = \frac{1}{\pi} P.V. \int \frac{f(x-\alpha) - f(x)}{\alpha}\, d\alpha.$$

In particular, one may easily check that $\partial_x \mathscr{H} = \Lambda$.

As for T, which is the nonlinear term, it is defined by

$$T(f) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\partial_x f(x) - \partial_x f(x-\alpha)}{\alpha} \frac{\left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2}{1 + \left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2} d\alpha. \tag{1.5}$$

Equivalently, the Muskat equation can be written as

$$(\mathcal{M}) \ : \ \begin{cases} \partial_t f = \dfrac{\rho}{\pi} P.V.\, \partial_x \displaystyle\int \arctan \Delta_\alpha f \ d\alpha \\ f(x,0) = f_0(x), \end{cases}$$

where $\Delta_\alpha f \equiv \frac{f(x,t)-f(x-\alpha,t)}{\alpha} = \frac{\delta_\alpha f(x,t)}{\alpha}$.

Indeed, it is well known that linearizing $\mathcal{M}$ around the flat solution gives rise to the fractional heat equation $f_t = \rho\Lambda f$ (see e.g., [1] and [14]). The equation is linearly stable if and only if the heavier fluid is below the interface (that is $\rho_2 > \rho_1$), otherwise we say that the curve is in the unstable regime. This is known as the Rayleigh-Taylor condition and is determined by the normal component of the pressure gradient jump at the interface having a distinguished sign (also called the Saffman-Taylor condition).

This equation has attracted the attention of the mathematical community in the past several years and we shall briefly sum up the results known regarding the Cauchy problem for $(\mathcal{M})$ in the stable regime ($\rho > 0$). First of all, let us recall that this equation has a maximum principle for $\|f(\cdot,t)\|_{L^\infty}$ and $\|f(\cdot,t)\|_{L^2}$. Indeed, it is shown in [15] that

$$\|f(t)\|_{L^\infty(\mathbb{R})} \leq \frac{\|f_0\|_{L^\infty(\mathbb{R})}}{1+t}.$$

Moreover, the authors showed in [15] that if $\|\partial_x f_0\|_{L^\infty} < 1$, then $\|\partial_x f(\cdot,t)\|_{L^\infty} < \|\partial_x f_0\|_{L^\infty}$ for all $t > 0$. On the other hand, there is also an L^2 maximum principle (see [12]). More precisely we have

$$\|f(T)\|^2_{L^2(\mathbb{R})} + \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2\right) d\alpha d\beta ds = \|f_0\|^2_{L^2(\mathbb{R})}.$$

This does not imply, for large initial data, a gain of regularity in the system. However, it was observed in [22] that a gain of regularity is possible even if we start with L^2 initial data, under the condition that the slope is initially less than 1 (see [22]). Recall that the Muskat equation has a scaling: if f is a solution associated to the initial data f_0, then the function $\lambda^{-1} f(\lambda x, \lambda t)$, $\lambda > 0$ is also a solution for the corresponding initial data $\lambda^{-1} f_0(\lambda x)$. In particular, the Lipschitz norm $\dot{W}^{1,\infty}$ is critical as well as the homogeneous Sobolev norm $\dot{H}^{3/2}$. More generally, the whole family of homogeneous Besov spaces $\dot{B}^{1+p^{-1}}_{p,q}$ with $(p,q) \in [1,+\infty]^2$ is critical with respect to the scaling of $\mathcal{M}$.

As far as the local well-posedness results are concerned, in [14], the authors proved local existence in H^3. The authors of [10] were able to lower the local theory to H^2. Recently, in [13], Constantin, Gancedo, Shvydkoy and Vicol have proved that the equation is locally well-posed in $\dot{W}^{2,p}$ with $p > 1$. There is another result by Matioc [26] where local existence is obtained in H^s, $s \in (3/2, 2]$. Instant analyticity is obtained in [9] from any initial data in H^4 (see also [26]).

If the heavier fluid is above the interface, *i.e.*, $\rho < 0$, then the equation $(\mathcal{M})$ is ill-posed in Sobolev spaces (see [14] and [3]). However, there exists weak solutions to the (IPM) system starting with an initial data with a jump of densities in the unstable regime. These solutions create a zone around the initial interface where the two fluids mix. This zone grows over time, for more details see [28], [33], [7] and [21].