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ANALYTIC NORMAL FORMS AND INVERSE PROBLEMS FOR UNFOLDINGS OF 2-DIMENSIONAL SADDLE-NODES WITH ANALYTIC CENTER MANIFOLD

BY C. ROUSSEAU AND L. TEYSSIER

ABSTRACT. – We give normal forms for generic k-dimensional parametric families $(Z_\varepsilon)_\varepsilon$ of germs of holomorphic vector fields near $0 \in \mathbb{C}^2$ unfolding a saddle-node singularity Z_0, under the condition that there exists a family of invariant analytic curves unfolding the weak separatrix of Z_0. These normal forms provide a moduli space for these parametric families. In our former 2008 paper, a modulus of a family was given as the unfolding of the Martinet-Ramis modulus, but the realization part was missing. We solve the realization problem in that partial case and show the equivalence between the two presentations of the moduli space. Finally, we completely characterize the families which have a modulus depending analytically on the parameter. We provide an application of the result in the field of non-linear, parameterized differential Galois theory.

RÉSUMÉ. – Nous donnons des formes normales pour les familles génériques $(Z_\varepsilon)_\varepsilon$ à k paramètres de germes de champs de vecteurs holomorphes au voisinage de $0 \in \mathbb{C}^2$, et déployant une singularité Z_0 de type col-nœud, sous la condition qu'il existe une famille de courbes analytiques invariantes déployant la séparatrice faible de Z_0. Ces formes normales donnent un espace de modules pour ces familles génériques. Dans notre article de 2008, nous avions donné un module de classification pour ces familles génériques, lequel consistait en un déploiement du module de Martinet-Ramis, mais la partie réalisation était manquante. Dans cet article, nous donnons la réalisation dans ce cas spécial, et nous montrons l'équivalence entre les deux présentations de l'espace des modules. Finalement, nous caractérisons complètement les familles dont le module dépend analytiquement des paramètres. Nous donnons une application du résultat en théorie de Galois paramétrique non linéaire.

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1. Introduction

Heuristically, moduli spaces of holomorphic dynamical systems not only encode but also describe qualitatively the dynamics itself, and to some extent allow a better understanding of remarkable dynamical phenomena. This paper is part of a large program aimed at studying the conjugacy classes of dynamical systems in the neighborhood of stationary points (up to local changes of analytic coordinates). Stationary points and their invariant manifolds organize the global dynamics while degenerate stationary points organize the bifurcation diagrams in families of dynamical systems. Stationary points of discrete dynamical systems correspond to fixed-points of the iterated map(s), while for continuous dynamical systems they correspond to singularities in the underlying differential equation(s).

A natural tool for studying conjugacy classes is the use of normal forms. For hyperbolic stationary points (generic situation), the system is locally conjugate to its linear part so that the quotient space of (local) hyperbolic systems is given by the space of linear dynamical systems. However, for most non-hyperbolic stationary points the normalizing change of coordinates (sending *formally* the system to a normal form) is given by a divergent power series. Divergence is very instructive: it tells us that the dynamics of the original system and that of the normal form are qualitatively different. In that respect, a subclass of singularities that has been thoroughly studied in the beginning of the 80's is that of 1-resonant singularities: these include parabolic fixed-points of germs of 1-dimensional diffeomorphisms, resonant-saddle singularities and saddle-node singularities of 2-dimensional vector fields, as well as non-resonant irregular singular points of linear differential systems. These various resonant dynamical systems share a lot of common properties, among which is the finite-determinacy of their formal normal forms (e.g., polynomial expressions in the case of vector fields). Another property they share is that they can be understood as the *coalescence* of special "geometric objects," either of stationary points or of a singular point with a limit cycle in the case of the Hopf bifurcation at a weak focus.

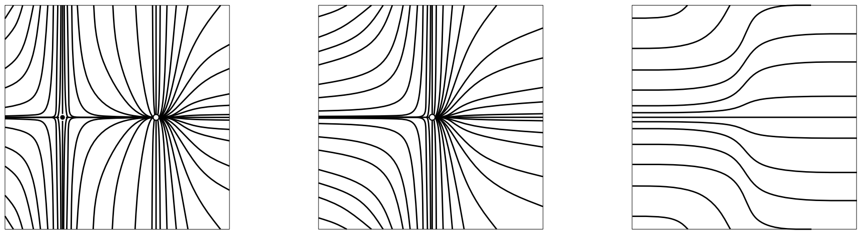
1.1. Scope of the paper

The present work is the follow-up of [41] in which we described a set of functional moduli for unfoldings of codimension k saddle-node vector fields $Z = (Z_\varepsilon)_\varepsilon$ depending on a finite-dimensional parameter $\varepsilon \in (\mathbb{C}^k, 0)$. Here we focus mainly on the inverse problem and on the question of finding (almost unique) normal forms, as we explain below.

The most basic example of such an unfolding is given by the codimension 1 unfolding (expressed in the canonical basis of $\mathbb{C}^2$)

$$Z_\varepsilon(x, y) := \begin{bmatrix} x^2 + \varepsilon \\ y \end{bmatrix}, \quad \varepsilon \in \mathbb{C}. \tag{1.1}$$

Real slices of the phase-portraits are shown in Figure 1.1. The merging (bifurcation) occurs at $\varepsilon = 0$: for $\varepsilon \neq 0$ the system has two stationary points located at $(\pm\sqrt{-\varepsilon}, 0)$ which collide as ε reaches 0.



(A) $\varepsilon < 0$ (B) $\varepsilon = 0$ (C) $\varepsilon > 0$

FIGURE 1.1. Typical members of the simplest saddle-node bifurcation.

1.2. Modulus of classification

Each merging stationary point organizes the dynamics in its own neighborhood in a rigid way. The local models of these rigid dynamics seldom agree on overlapping areas and in general cannot be glued together. If this incompatibility persists as the confluence happens, then we have divergence of the normalizing series at the limit. In the case of 1- or 2-dimensional resonant systems the normalizing series is k-summable. The divergence is then quantified by the *Stokes phenomenon*: there exists a formal normalizing transformation, and a covering of a punctured neighborhood of the singularity by $2k$ sectors over which there exist unique sectorial normalizing transformations that are Gevrey-asymptotic to the formal normalization. Comparing the normalizing transformations on intersections of consecutive sectors provides a modulus of analytic classification. This modulus takes the form of Stokes matrices for irregular singularities of linear differential systems and functional moduli for singularities of nonlinear dynamical systems (see for instance [20]).

The classification of resonant systems may seem rather mysterious. But if we remember that we are studying the merging of "simple" singularities, then it becomes natural to unfold the situation and study the "multiple" singularity as a limiting case. Indeed, analyzing unfoldings sheds a new light on the "complicated" dynamics of the limiting systems. The idea was suggested by several mathematicians, including V. Arnold, A. Bolibruch and J. Martinet [30]. It was put in practice for unfoldings of saddle-node singularities by A. Glutsyuk [15] on regions in parameter space over which the confluent singularities are all hyperbolic. The system can be linearized in the neighborhood of each singularity, and the mismatch in the normalizing changes of coordinates tends to the components of the saddle-node's Martinet-Ramis modulus [31] when the singularities merge. But the tools were still missing for a full classification of unfoldings of multiple singularities, in particular on a full neighborhood in parameter space of the bifurcation value.

The thesis of P. Lavaurs [24] on parabolic points of diffeomorphisms opened the way for such classifications, for he studied the complementary regions in parameter space. The first classification of generic unfoldings of codimension 1 fixed-point of diffeomorphisms regarded the parabolic point [29], and then the resonant-saddle and saddle-node singularities of differential equations [37, 38]. The first classification of generic unfoldings of codimension k saddle-nodes was done by the authors [41] using the visionary ideas of