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DISCONTINUITY OF THE PHASE TRANSITION FOR THE PLANAR RANDOM-CLUSTER AND POTTS MODELS WITH $q > 4$

BY HUGO DUMINIL-COPIN, MAXIME GAGNEBIN, MATAN HAREL, IOAN MANOLESCU AND VINCENT TASSION

ABSTRACT. – We prove that the q-state Potts model and the random-cluster model with cluster weight $q > 4$ undergo a discontinuous phase transition on the square lattice. More precisely, we show (1) Existence of multiple infinite-volume measures for the critical Potts and random-cluster models, (2) Ordering for the measures with monochromatic (resp. wired) boundary conditions for the critical Potts model (resp. random-cluster model), and (3) Exponential decay of correlations for the measure with free boundary conditions for both the critical Potts and random-cluster models. The proof is based on a rigorous computation of the Perron-Frobenius eigenvalues of the diagonal blocks of the transfer matrix of the six-vertex model, whose ratios are then related to the correlation length of the random-cluster model.

As a byproduct, we rigorously compute the correlation lengths of the critical random-cluster and Potts models, and show that they behave as $\exp(\pi^2/\sqrt{q-4})$ as q tends to 4.

RÉSUMÉ. – Nous démontrons que la transition de phase du modèle de Potts à q états et de la percolation FK avec $q > 4$ est du premier ordre. Plus précisément, nous montrons: (1) l'existence de plusieurs mesures en volume infini pour ces modèles au point critique, (2) l'émergence d'une structure ordonnée pour les mesures avec conditions au bord monochromatiques (resp. liées) pour le modèle de Potts critique (resp. pour la percolation FK), et (3) la décroissance exponentielle des corrélations pour les mesures libres des deux modèles au point critique. La preuve repose sur un calcul rigoureux des valeurs propres de Perron Frobenius associées aux blocs diagonaux de la matrice de transfert du modèle « six-vertex », qui peuvent être directement reliées à la longueur de corrélation de la percolation FK. Notamment, cette approche nous donne un calcul rigoureux des longueurs de corrélation critiques pour la percolation FK et le modèle de Potts au point critique. Nous en déduisons un comportement asymptotique de la forme $\exp(\pi^2/\sqrt{q-4})$ lorsque le paramètre q tend vers 4.

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1. Introduction

1.1. Motivation

Lattice spin models were introduced to describe specific experiments; they were later found to be illustrative of a large variety of physical phenomena. Depending on a parameter (most commonly temperature), they exhibit different macroscopic behaviors (also called phases), and phase transitions between them. Phase transitions may be continuous or discontinuous, and determining their type is one of the first steps towards a deeper understanding of the model.

In recent years, the Potts and random-cluster models have been the object of revived interest after new rigorous results were proved. In [3], the critical points of the models were determined for any $q \geq 1$. In [12], the models were proved to undergo a continuous phase transition for $1 \leq q \leq 4$, thus proving half of a famous prediction by Baxter. The object of this paper is to prove the second half of his prediction - namely, that the phase transition is discontinuous when $q > 4$.

1.2. Results for the Potts model

The Potts model was introduced by Potts [21] following a suggestion of his adviser Domb. While the model received little attention early on, it became the object of great interest in the last 50 years. Since then, mathematicians and physicists have been studying it intensively, and much is known about its rich behavior, especially in two dimensions. For a review of the physics results, see [24].

In this paper, we will focus on the case of the square lattice $\mathbb{Z}^2$ composed of vertices $x = (x_1, x_2) \in \mathbb{Z}^2$, and edges between nearest neighbors. In the q-state ferromagnetic Potts model (where q is a positive integer larger than or equal to 2), each vertex of a graph receives a *spin* taking value in $\{1, \ldots, q\}$. The energy of a configuration is then proportional to the number of neighboring vertices of the graph having different spins. Formally, the Potts measure on a finite subgraph $G = (V, E)$ of the square lattice, at inverse temperature $\beta > 0$ and boundary conditions $i \in \{0, 1, \ldots, q\}$, is defined for every $\sigma \in \{1, \ldots, q\}^V$ by the formula

$$\mu^i_{G,\beta}[\sigma] := \frac{\exp[-\beta \mathbf{H}^i_G(\sigma)]}{\displaystyle\sum_{\sigma' \in \{1,\ldots,q\}^V} \exp[-\beta \mathbf{H}^i_G(\sigma')]}, \tag{1.1}$$

where

$$\mathbf{H}^i_G(\sigma) := -\sum_{\{x,y\} \in E} \mathbf{1}[\sigma_x = \sigma_y] - \sum_{x \in \partial V} \mathbf{1}[\sigma_x = i].$$

Above, $\mathbf{1}[\cdot]$ denotes the indicator function and ∂V is the set of vertices of G with at least one neighbor (in $\mathbb{Z}^2$) outside of G. Note that when $i = 0$, the second sum is zero for all σ.

For any boundary conditions i, the family of measures $\mu^i_{G,\beta}$ converges as G tends to the whole square lattice. The resulting measure μ^i_β defined on the square lattice is called the Gibbs measure with *free* boundary conditions if $i = 0$ (respectively, *monochromatic* boundary conditions equal to i if $i \in \{1, \ldots, q\}$).

The Potts model undergoes an order/disorder phase transition, meaning that there exists a *critical inverse temperature* $\beta_c = \beta_c(q) \in (0, \infty)$) such that:

— For $\beta < \beta_c$, the measures μ^i_β, $i = 0, \dots, q$, are all equal.

— For $\beta > \beta_c$, the measures μ^i_β, $i = 0, \dots, q$, are all distinct.

Baxter [1] conjectured that the phase transition is continuous if $q \leq 4$ and discontinuous if $q > 4$, meaning that all the measures $\mu^i_{\beta_c}$ with $i = 0, \dots, q$ are equal if and only if $q \leq 4$. It was shown in [3] that $\beta_c = \log(1 + \sqrt{q})$; moreover, when $q \leq 4$, it was proved in [12] that the phase transition is indeed continuous, along with more detailed properties of the unique critical measure μ_{β_c}. The goal of this article is to complete the proof of Baxter's conjecture by proving the following theorem. Below, x_n denotes the site of $\mathbb{Z}^2$ with both coordinates equal to $\lfloor n/2 \rfloor$.

THEOREM 1.1. – *Consider the q-state Potts model on the square lattice with $q > 4$. Then,*

1. *all the measures $\mu^i_{\beta_c}$ for $i = 0, \dots, q$ are distinct and ergodic (in particular, $\mu^0_{\beta_c}$ is not equal to the average of the $\mu^i_{\beta_c}$ with $i \in \{1, \dots, q\}$);*
2. *for any $i \in \{1, \dots, q\}$, $\mu^i_{\beta_c}[\sigma_0 = i] > \frac{1}{q}$.*
3. *Let $\lambda > 0$ satisfy $\cosh(\lambda) = \sqrt{q}/2$. Then*

$$\lim_{n\to\infty} -\tfrac{1}{n} \log\left(\mu^0_{\beta_c}[\sigma_0 = \sigma_{x_n}] - \tfrac{1}{q}\right) = \lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda).$$

Furthermore, the quantity above is strictly positive.

The limit computed in the final item above is the inverse correlation length of the critical Potts model in the diagonal direction. This theorem follows directly from Theorem 1.2 below via the standard coupling between the Potts and random-cluster models (see Section 3.4 for details).

1.3. Results for the random-cluster model

The random-cluster model (also called Fortuin-Kasteleyn percolation) was introduced by Fortuin and Kasteleyn around 1970 (see [14] and [15]) as a class of models satisfying specific series and parallel laws. It is related to many other models of statistical mechanics, including the Potts model. For background on the random-cluster model and the results mentioned below, we direct the reader to the monographs [18] and [7].

Consider a finite subgraph $G = (V, E)$ of the square lattice. A percolation configuration ω is an element of $\{0, 1\}^E$. An edge e is said to be *open* (in ω) if $\omega(e) = 1$, otherwise it is *closed*. A configuration ω can be seen as a subgraph of G with vertex set V and edge-set $\{e \in E : \omega(e) = 1\}$. When speaking of connections in ω, we view ω as a graph. A *cluster* is a connected component of ω (it may just be an isolated vertex). Let $o(\omega)$ and $c(\omega)$ denote the number of open edges and closed edges in ω respectively. Let $k_0(\omega)$ denote the number of clusters of ω, and $k_1(\omega)$ the number of clusters of ω when all clusters intersecting ∂V are counted as a single one – as before, ∂V is the set of vertices of G adjacent to a vertex of $\mathbb{Z}^2$ not contained in G.