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TENSOR PRODUCT MULTIPLICITIES VIA UPPER CLUSTER ALGEBRAS

BY JIARUI FEI

ABSTRACT. – For each valued quiver Q of Dynkin type, we construct a valued ice quiver Δ_Q^2. Let G be a simply connected Lie group with Dynkin diagram the underlying valued graph of Q. The upper cluster algebra of Δ_Q^2 is graded by the triple dominant weights (μ, ν, λ) of G. We prove that when G is simply-laced, the dimension of each graded component counts the tensor multiplicity $c_{\mu,\nu}^{\lambda}$. We conjecture that this is also true if G is not simply-laced, and sketch a possible approach. Using this construction, we improve Berenstein-Zelevinsky's model, or in some sense generalize Knutson-Tao's hive model in type A.

RÉSUMÉ. – Nous construisons un carquois valué glacé Δ_Q^2 pour chaque carquois valué de type Dynkin. Soit G un groupe de Lie simplement connexe dont le diagramme de Dynkin est le graphe valué sous-jacent de Q. L'algèbre amassée supérieure de Δ_Q^2 est graduée par le triplet de poids dominants (μ, ν, λ) de G. Lorsque G est simplement lacé, nous montrons que la dimension de chaque composante graduée compte $c_{\mu,\nu}^{\lambda}$ la multiplicité tensorielle. Nous conjecturons que c'est aussi le cas lorsque G n'est pas simplement lacé, et nous esquissons une approche possible. En utilisant cette construction, nous améliorons le modèle de Berenstein-Zelevinsky, ou en un certain sens, nous généralisons le modèle de ruche de Knutson-Tao en type A.

Introduction

Finding the polyhedral model for the tensor multiplicities in Lie theory is a long-standing problem. By *tensor multiplicities* we mean the multiplicities of irreducible summands in the tensor product of any two finite-dimensional irreducible representations of a simply connected Lie group G. The problem asks to express the multiplicity as the number of lattice points in some convex polytope.

Accumulating from the works of Gelfand, Berenstein and Zelevinsky since 1970's, a first quite satisfying model for G of type A was invented in [4]. Finally around 1999, building

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upon their work, Knutson and Tao invented their *hive model*, which led to the solution of the *saturation conjecture* [35]. In fact, the reduction of Horn's problem to the Saturation conjecture is an important driving force for the evolution of the models.

Outside type A, up to now Berenstein and Zelevinsky's models [5] are still the only known polyhedral models. Those models lose a few nice features of Knutson-Tao's hive model. We will have a short discussion on this in Section 0.1. Despite a lot of effort to improve the Berenstein-Zelevinsky model, to the author's best knowledge there is no very satisfying further result in this direction.

Recently an interesting link between the hive model and the *cluster algebra* theory was established in [13] through the Derksen-Weyman-Zelevinsky's *quiver with potential* model [8, 9] for cluster algebras. A similar but different link between the polyhedral models and tropical geometry was established by Goncharov and Shen in [29]. In fact, from the work of Berenstein, Fomin and Zelevinsky [5, 3], those links may not be a big surprise.

There are two goals in the current paper. First we want to generalize the work [13] to other types. More specifically, we hope to prove that the algebras of regular functions on certain configuration spaces are all upper cluster algebras. Second we want to improve the Berenstein-Zelevinsky's model in the spirit of Knutson-Tao. In fact, as we shall see, we accomplish these two goals almost simultaneously. Namely, we use our conjectural models to establish the cluster algebra structures. Once the cluster structures are established, the conjectural models are proved as well.

The key to making new models is the construction of the *iARt quivers*. Let Q be a valued quiver of Dynkin type. Let C^2Q be the category of projective presentations of Q. We can associate to this category an *Auslander-Reiten quiver* $\Delta(C^2Q)$ with translation (ARt quiver in short). The ice ARt quiver (iARt quiver in short) Δ_Q^2 is obtained from $\Delta(C^2Q)$ by *freezing* three sets of vertices, which correspond to the *negative*, *positive*, and *neutral* presentations in C^2Q. We can put a (quite canonical) *potential* W_Q^2 on the iARt quiver Δ_Q^2.

A quiver with potential (or QP in short) (Δ, W) is related to Berenstein-Fomin-Zelevinsky's *upper cluster algebras* [3] through *cluster characters* evaluating on *μ-supported* g-*vectors* introduced in [13] (see Definition 4.5 and 4.8). The cluster character C_W considered in this paper is the *generic* one [42, 12], but it can be replaced by fancier ones. As we have seen in many different situations [13, 14, 15] the set $G(\Delta, W)$ of μ-supported g-vectors is given by lattice points in some rational polyhedral cone. This is also the case for the iARt QPs (Δ_Q^2, W_Q^2).

The whole Part I is devoted to the construction of the iARt QP (Δ_Q^2, W_Q^2) and the polyhedral cone $\mathsf{G}_{\Delta_Q^2}$. It turns out that the cone $\mathsf{G}_{\Delta_Q^2}$ has a very neat hyperplane presentation $\{x \in \mathbb{R}^{(\Delta_Q^2)_0} \mid xH \geq 0\}$, where the columns of the matrix H are given by the dimension vectors of subrepresentations of $3|Q_0|$ representations of Δ_Q^2. These $3|Q_0|$ representations are in bijection with the frozen vertices of Δ_Q^2. They also have a very simple and nice description (see Theorem 5.3). The main result of Part I is the following.

THEOREM 5.9. – *The set* $\mathsf{G}_{\Delta_Q^2} \cap \mathbb{Z}^{(\Delta_Q^2)_0}$ *is exactly* $G(\Delta_Q^2, W_Q^2)$.

The upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^2)$ has a natural grading by the *weight vectors* of presentations. This grading can be extended to a triple-weight grading $\boldsymbol{\sigma}_Q^2 : \mathbb{Z}^{(\Delta_Q^2)_0} \to \mathbb{Z}_{\geqslant 0}^{3|Q_0|}$. This grading slices the cone $\mathsf{G}_{\Delta_Q^2}$ into polytopes

$$\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda) := \left\{ \mathsf{g} \in \mathsf{G}_{\Delta_Q^2} \mid \boldsymbol{\sigma}_Q^2(\mathsf{g}) = (\mu,\nu,\lambda) \right\}.$$

Let $G := G_Q$ be the simply connected simple Lie group with Dynkin diagram the underlying valued graph of Q. Our conjectural model is that the lattice points in $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda)$ count the tensor multiplicity $c_{\mu\,\nu}^{\lambda}$ for G. Here, $c_{\mu\,\nu}^{\lambda}$ is the multiplicity of the irreducible representation $L(\lambda)$ of highest weight λ in the tensor product $L(\mu) \otimes L(\nu)$. More often than not we identify a dominant weight by a non-negative integral vector. To prove this model, we follow a similar line as [13]. However, we do not have a quiver setting to work with in general. We replace the semi-invariant rings of triple-flag quiver representations by the ring of regular functions on a certain configuration space introduced in [19].

Fix an opposite pair of maximal unipotent subgroups (U^-, U) of G. The quotient space $\mathcal{A} := U^- \backslash G$ is called *base affine space*, and the quotient space $\mathcal{A}^\vee := G/U$ is called its *dual*. The configuration space $\mathrm{Conf}_{2,1}$ is by definition $(\mathcal{A} \times \mathcal{A} \times \mathcal{A}^\vee)/G$, where G acts multi-diagonally. The ring of regular functions $k[\mathrm{Conf}_{2,1}]$ is just the invariant ring $(k[G]^{U^-} \otimes k[G]^{U^-} \otimes k[G]^{U})^G$. The ring $k[\mathrm{Conf}_{2,1}]$ is multigraded by a triple of weights (μ,ν,λ). Each graded component $C_{\mu,\nu}^{\lambda} := k[\mathrm{Conf}_{2,1}]_{\mu,\nu,\lambda}$ is given by the G-invariant space $(L(\mu) \otimes L(\nu) \otimes L(\lambda)^\vee)^G$. So the dimension of $C_{\mu,\nu}^{\lambda}$ counts the tensor multiplicity $c_{\mu\,\nu}^{\lambda}$. Here is the main result of Part II.

THEOREM 9.1. – *Suppose that Q is trivially valued. Then the ring of regular functions on* $\mathrm{Conf}_{2,1}$ *is the graded upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2; \boldsymbol{\sigma}_Q^2)$. Moreover, the generic character maps the lattice points in $\mathsf{G}_{\Delta_Q^2}$ onto a basis of this algebra. In particular, $c_{\mu\,\nu}^{\lambda}$ is counted by lattice points in $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda)$.*

We will show by an example that the upper cluster algebra strictly contains the corresponding cluster algebra in general. We conjecture that the trivially valued assumption can be dropped in the above theorem and the theorem below. It is pointed in the end that the only missing ingredient for proving the conjecture is the analogue of [9, Lemma 5.2] for *species with potentials* [37].

Fock and Goncharov studied in [19] the similar spaces Conf_3 [(1)] as cluster varieties. However, to the author's best knowledge it is not clear from their discussion what an *initial seed* is if G is not of type A. Moreover the equality established in the theorem does not seem to follow from any result there. In fact, Fock and Goncharov later conjectured in [20] that the tropical points in their cluster $\mathcal{X}$-varieties parametrize bases in the corresponding (upper) cluster algebras. Our result can be viewed as an algebraic analog of their conjecture for the space $\mathrm{Conf}_{2,1}$. Instead of working with the tropical points, we work with the g-vectors.

To sketch our ideas, we first observe that if we forget the frozen vertices corresponding to the positive and neutral presentations, then we get a valued ice quiver denoted by Δ_Q

(1) They considered the generic part of the quotient stack $[(\mathcal{A}^\vee)^3/G]$. We will work with the categorical quotient as its partial compactification.