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ARITHMETIC AMPLENESS AND AN ARITHMETIC BERTINI THEOREM

BY FRANÇOIS CHARLES

ABSTRACT. – Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2. If $\overline{\mathcal{L}}$ is an ample hermitian line bundle on $\mathcal{X}$, we prove that the proportion of those effective sections σ of $\overline{\mathcal{L}}^{\otimes n}$ such that the divisor of σ on $\mathcal{X}$ is irreducible tends to 1 as n tends to ∞. We prove variants of this statement for schemes mapping to such an $\mathcal{X}$.

On the way to these results, we discuss some general properties of arithmetic ampleness, including restriction theorems, and upper bounds for the number of effective sections of hermitian line bundles on arithmetic varieties.

RÉSUMÉ. – Soit $\mathcal{X}$ une variété arithmétique projective de dimension au moins 2, et soit $\overline{\mathcal{L}}$ un fibré hermitien sur $\mathcal{X}$. Si $\overline{\mathcal{L}}$ est ample, on démontre que la proportion des sections effectives de $\overline{\mathcal{L}}^n$ qui définissent un diviseur irréductible sur $\mathcal{X}$ tend vers 1 quand n tend vers ∞. On démontre également des variantes de ce résultat pour des schémas admettant un morphisme vers $\mathcal{X}$.

On prouve par ailleurs un certain nombre de propriétés générales de l'amplitude arithmétique, autour notamment de théorèmes de restriction et d'estimées pour le nombre de sections effectives de fibrés en droites hermitiens.

1. Introduction

1.1. Bertini theorems over fields

Let k be an infinite field, and let X be an irreducible variety over k with dimension at least 2. Given an embedding of X in some projective space over k, the classical Bertini theorem [23, Theorem 3.3.1] shows, in its simplest form, that infinitely many hyperplane sections of X are irreducible.

In the case where k is finite, the Bertini theorem can fail, since the finitely many hyperplane sections of X can all be reducible. As was first explained in [26] in the setting of smoothness theorems, this phenomenon can be dealt with by replacing hyperplane sections with ample hypersurfaces of higher degree. We can state the main result of [11]—see Theorem 1.6 in [11] and the discussion in the proof of Theorem 6.1 below—as follows: let k be a finite field, let

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X be a projective variety over k and let L be an ample line bundle on X. Let Y be an integral scheme of finite type over k together with a morphism $f : Y \to X$. Assume that the image of f has dimension at least 2. If Z is a subscheme of Y, write Z_{horiz} for the union of those irreducible components of Z that do not map to a closed point of X. Then the set

$$\mathcal{P} = \left\{\sigma \in \bigcup_{n>0} H^0(X, L^{\otimes n}),\ \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\right\}$$

has density 1, in the sense that

$$\lim_{n\to\infty} \frac{|\mathcal{P} \cap H^0(X, L^{\otimes n})|}{|H^0(X, L^{\otimes n})|} = 1.$$

Here if S is a set, we denote by $|S|$ its cardinality. When Y is a subscheme of X, we can disregard the horizontality subscript.

1.2. The arithmetic case

In this paper, we deal with an arithmetic version of Bertini theorems as above. Let $\mathcal{X}$ be an arithmetic variety, that is, an integral scheme which is separated, flat of finite type over $\mathrm{Spec}\,\mathbb{Z}$. Assume that $\mathcal{X}$ is projective, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$. As is well known, sections of $\mathcal{L}$ over $\mathcal{X}$ are not the analogue of global sections of a line bundle over a projective variety over a field. Indeed, it is more natural to consider a hermitian line bundle $\overline{\mathcal{L}}$ with underlying line bundle $\mathcal{L}$ and consider the sets

$$H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$$

of sections with norm at most 1 everywhere. We discuss ampleness for hermitian line bundles in Section 2, which we refer to for definitions.

Given finite sets $(X_n)_{n>0}$, and a subset $\mathcal{P}$ of $\bigcup_{n>0} X_n$, say that $\mathcal{P}$ has density ρ if the following equality holds:

$$\lim_{n\to\infty} \frac{|\mathcal{P} \cap X_n|}{|X_n|} = \rho.$$

The main result of this paper is the following arithmetic Bertini theorem. Again, given a morphism of schemes $f : Y \to X$, and if Z is a subscheme of Y, we denote by Z_{horiz} the union of those irreducible components of Z that do not map to a closed point of X. If $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ is a hermitian vector bundle and δ is a real number, write $\overline{\mathcal{M}}(\delta)$ for the hermitian vector bundle $(\mathcal{M}, e^{-\delta}||.||)$. Write $||\sigma||_\infty$ for the sup norm of a section of a hermitian vector bundle.

THEOREM 1.1. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be an integral scheme of finite type over* $\mathrm{Spec}\,\mathbb{Z}$ *together with a morphism $f : \mathcal{Y} \to \mathcal{X}$ which is generically smooth over its image. Assume that the image of $\mathcal{Y}$ has dimension at least* 2. *Let ε be a positive real number. Then the set*

$$\left\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}),\ ||\sigma|_{f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1 \text{ and } \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\right\}$$

has density 1 *in* $\left\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}),\ ||\sigma|_{f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1\right\}$.

Recall that by definition, the condition $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n})$ means

$$||\sigma||_\infty \leq \varepsilon^n.$$

REMARK 1.2. – A Weil divisor is said to be irreducible if it comes from an integral codimension 1 subscheme.

REMARK 1.3. – The hypothesis that f is generically smooth over its image is necessary: when f is the Frobenius morphism of a fiber $\mathcal{X}_p$, all $\mathrm{div}(f^*\sigma)$ have components with multiplicities divisible by p. Of course, it holds when $\mathcal{Y}$ is flat over $\mathrm{Spec}\,\mathbb{Z}$. Without this hypothesis on f, the conclusion is only that the support of $\mathrm{div}(f^*\sigma)$ is irreducible for a density 1 set of σ.

An important special case of the theorem deals with the special case where f is dominant. In this case, generic smoothness is automatic.

THEOREM 1.4. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be an integral scheme of finite type over* $\mathrm{Spec}\,\mathbb{Z}$ *together with a morphism $f : \mathcal{Y} \to \mathcal{X}$. Assume that the image of $\mathcal{Y}$ has dimension at least* 2 *and f is dominant. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\Big\}$$

has density 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

REMARK 1.5. – We will prove Theorem 1.1 as a consequence of Theorem 1.4. However, the latter is a special case of the former. Indeed, with the notation of Theorem 1.1, when f is dominant, if $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}$, then the condition

$$||\sigma|_{f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1$$

is equivalent to

$$||\sigma||_\infty \leq 1,$$

i.e., $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

The case where $\mathcal{Y} = \mathcal{X}$ is particularly significant. We state it independently below. Most of this paper will be devoted to its proof, and we will prove 1.1 and 1.4 as consequences.

THEOREM 1.6. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least* 2, *and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \mathrm{div}(\sigma) \text{ is irreducible}\Big\}$$

has density 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.