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CONVERGENCE OF THE SELF-AVOIDING WALK ON RANDOM QUADRANGULATIONS TO $SLE_{8/3}$ ON $\sqrt{8/3}$-LIOUVILLE QUANTUM GRAVITY

BY Ewain GWYNNE AND Jason MILLER

ABSTRACT. – We prove that a uniform infinite quadrangulation of the half-plane decorated by a self-avoiding walk (SAW) converges in the scaling limit to the metric gluing of two independent Brownian half-planes identified along their positive boundary rays. Combined with other work of the authors, this implies the convergence of the SAW on a random quadrangulation to $SLE_{8/3}$ on a certain $\sqrt{8/3}$-Liouville quantum gravity surface. The topology of convergence is the local Gromov-Hausdorff-Prokhorov-uniform topology, the natural generalization of the local Gromov-Hausdorff topology to curve-decorated metric measure spaces. We also prove analogous scaling limit results for uniform infinite quadrangulations of the whole plane decorated by either a one-sided or two-sided SAW. Our proof uses only the peeling procedure for random quadrangulations and some basic properties of the Brownian half-plane, so that it can be read without any knowledge of SLE or LQG.

RÉSUMÉ. – Nous considérons une configuration consistant en une quadrangulation aléatoire infinie uniforme du demi-plan, décorée par un chemin auto-évitant infini, également aléatoire. Nous montrons dans cet article que dans la limite d'échelle, la loi de cette configuration converge vers le recollement métrique de deux demi-plans Browniens le long de leurs demi-droites positives (les demi-plans Browniens sont des métriques aléatoires définies sur le demi-plan).

En combinant ce résultat avec ceux de nos articles antérieurs, ceci complète la preuve de la convergence en loi de ces configurations vers celle d'un $SLE_{8/3}$ dessiné sur une surface aléatoire dite de gravité quantique de Liouville de paramètre $\sqrt{8/3}$. Nous établissons également le résultat analogue pour des quadrangulations uniformes du plan (au lieu du demi-plan) décoré par des marches auto-évitantes infinies ou bi-infinies.

Ces convergences sont établies pour la topologie de Gromov-Hausdorff-Prohorov, qui est l'extension naturelle de la topologie de Gromov-Hausdorff pour ce type de configurations consistant en un espace métrique décoré par une courbe.

Notre preuve utilise la procédure d'exploration des quandrangulations aléatoires par épluchage et certaines propriétés de base du demi-plan Brownien, de sorte que le présent article ne demande aucun prérequis sur les processus SLE ou la gravité quantique de Liouville.

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1. Introduction

1.1. Overview

1.1.1. *Self-avoiding walk.* – Suppose that $G = (\mathcal{V}(G), \mathcal{E}(G))$ is a graph and $x, y \in \mathcal{V}(G)$ are distinct vertices. The *self-avoiding walk* (SAW) on G from x to y of length n is the uniform measure on paths from x to y in G of length n which do not self-intersect. The SAW was first introduced as a model for a polymer by Flory [27]. There is a vast literature on the SAW in both mathematics and physics and we will not attempt to survey it in its entirety here, but we will now briefly mention a few of the basic results which are most closely related to the present work.

The first question that one is led to ask about the SAW is *how many are there*? If G is an infinite, vertex transitive graph (such as $\mathbb{Z}^d$) and c_n denotes the number of SAWs in G starting from a given vertex with length n, then it is not difficult to see that $c_{m+n} \leq c_m c_n$ for each $m, n \in \mathbb{N}$. Consequently, the limit $\mu = \lim_{n\to\infty} c_n^{1/n}$ exists and is the so-called *connective constant* [34]. There is an extensive literature on the connective constant for various graphs. See, e.g., the survey provided in [28, 29] and the references therein. We mention that the connective constant in the case of the two-dimensional hexagonal lattice was shown to be $\sqrt{2+\sqrt{2}}$ in [21], but identifying this constant for other lattices remains an open problem.

The next natural question that one is led to ask is whether the SAW possesses a *scaling limit*, and this is the question which will be the focus of the present work. Building on work of Brydges and Spencer [15], it was shown by Hara and Slade that the SAW on the integer lattice in dimension $d \geq 5$ converges to Brownian motion when one performs a diffusive scaling [35]. The scaling limit of the SAW is also conjectured to be given by Brownian motion when $d = 4$, but with an extra logarithmic correction in the scaling. This has not yet been proved, although a number of theorems about weakly self-avoiding walk have been proven; see [5] for a recent survey. It is not known what the scaling limit (or factor) should be for $d = 3$, though various exponents associated with this case have been derived numerically. We refer to the survey articles [60, 6, 29] and the book [42] and the references therein for more results on the SAW.

The main focus of the present work is the case $d = 2$. It was conjectured by Lawler, Schramm, and Werner [39] that in this case the SAW converges upon appropriate rescaling to the Schramm-Loewner evolution (SLE) [56] with parameter $\kappa = 8/3$. This conjecture was derived by making the ansatz that the scaling limit of the SAW should be conformally invariant and satisfy a certain Markov property. The value $\kappa = 8/3$ arises because $\mathrm{SLE}_{8/3}$ satisfies the so-called *restriction property* [38], which is the continuum analog of the fact that a SAW conditioned to stay in a subgraph is the same as a SAW on that subgraph. This conjecture has been supported by extensive numerical simulations due to Tom Kennedy [36]. Prior to the present work, no scaling limit result for the SAW in two dimensions has been proved, however.

We will study and prove scaling limit results for the SAW in two dimensions on certain types of *random planar maps*. The SAW in this context was first studied (non-rigorously) by Duplantier and Kostov [22, 23] as a test case for the KPZ formula [37], which relates exponents for critical models on random surfaces with the corresponding exponents on planar lattices. We will establish the existence of the scaling limit of the SAW on a *random* *planar quadrangulation*, viewed as a curve-decorated metric measure space equipped with the SAW, the graph distance, and the counting measure on edges. Although the proof of this scaling limit result uses only the theory of random planar maps, the results of [32] allow us to identify the limiting object with $SLE_{8/3}$ on a $\sqrt{8/3}$-*Liouville quantum gravity wedge*, a certain random metric measure space with the topology of the upper half-plane. We will discuss this identification further in Section 1.1.4, but first let us say more about SAW on random planar maps.

Recall that a planar map is a graph together with an embedding in the plane so that no two edges cross. Two such maps are said to be equivalent if there exists an orientation preserving homeomorphism which takes one to the other. A map is said to be a quadrangulation if every face has exactly four adjacent edges.

The theories of statistical mechanics models like the SAW on random planar maps and on deterministic lattices are equally important: both are well-motivated physically and have been studied extensively in the math and physics literature. There are many questions (such as scaling limits of various curves toward SLE) which can be asked in both the random planar map and deterministic lattice settings (in the former setting, one has to specify a topology). It is not in general clear which setting is easier to analyze.

The convergence of the SAW toward $SLE_{8/3}$ is particularly interesting since in both the random planar map and deterministic lattice settings, the SAW is easy to define and important both mathematically and physically; the convergence toward $SLE_{8/3}$ is supported by heuristic evidence; and, prior to this work, the convergence was not proven rigorously in either setting.

1.1.2. *Gluing together random quadrangulations*. – We will now describe a simple construction of a finite quadrangulation decorated with a SAW and then describe the corresponding infinite volume versions of this construction.

Suppose we sample two independent uniformly random quadrangulations of the disk with simple boundary with n quadrilaterals and perimeter $2l$ and then glue them together along a boundary segment of length $2s < 2l$ by identifying the corresponding edges (Figure 1, left). The conditional law of the gluing interface given the overall glued map will then be that of a SAW of length $2s$ conditioned on its left and right complementary components both containing n quadrilaterals. One can also glue the *entire* boundaries of the two disks to obtain a map with the topology of the sphere decorated by a path whose conditional law given the map is that of a self-avoiding loop on length $2l$ conditioned on the two complementary components both containing n quadrilaterals. See, for example, the discussion in [11, Section 8.2] (which builds on [14, 13]) for additional explanation.

The *uniform infinite half-planar quadrangulation with simple boundary* ($UIHPQ_S$) is the infinite-volume local limit of uniform quadrangulations of the disk with simple boundary rooted at a boundary edge as the total number of interior faces (or interior vertices), and then the number of boundary edges, is sent to ∞ [20, 16].

It is shown by Caraceni and Curien [17, Section 1.4] that the infinite volume limit of the aforementioned random SAW-decorated quadrangulations can be constructed by starting off with two independent $UIHPQ_S$'s and then gluing them together along their boundary (Figure 1, right). In this case, the gluing interface is an infinite volume limit of a SAW. There