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ON THE VANISHING COHOMOLOGY PROBLEM FOR COCYCLE ACTIONS OF GROUPS ON II_1 FACTORS

BY Sorin POPA

Abstract. – We prove that any free cocycle action of a countable amenable group Γ on any II_1 factor N can be perturbed by inner automorphisms to a genuine action. Besides being satisfied by all amenable groups, this universal *vanishing cohomology* property, that we call $\mathcal{VC}$, is also closed to free products with amalgamation over finite groups. While no other examples of $\mathcal{VC}$-groups are known, by considering special cocycle actions $\Gamma \curvearrowright N$ in the case N is the hyperfinite II_1 factor R, respectively the free group factor $N = L(\mathbb{F}_\infty)$, we exclude many groups from being $\mathcal{VC}$. We also show that any free action $\Gamma \curvearrowright R$ gives rise to a free cocycle Γ-action on the II_1 factor $R' \cap R^\omega$ whose vanishing cohomology is equivalent to Connes' Approximate Embedding property for the II_1 factor $R \rtimes \Gamma$.

Résumé. – On démontre que toute action libre à cocycle d'un groupe dénombrable moyennable sur un facteur II_1 peut être perturbée par des automorphismes intérieurs de telle manière qu'elle devienne une « vraie » action. En plus de contenir tous les groupes moyennables, la classe des groupes qui satisfont cette propriété d'annulation en cohomologie, qu'on appelle $\mathcal{VC}$, est stable par produit libre amalgamé au-dessus d'un groupe fini. Il n'y a pas d'autres exemples connus de groupes $\mathcal{VC}$, mais en utilisant certaines actions à cocycle $\Gamma \curvearrowright N$ dans le cas où N est le facteur hyperfini R, respectivement N est le facteur du groupe libre $L(\mathbb{F}_\infty)$, on montre que beaucoup de groupes Γ ne sont pas $\mathcal{VC}$. On montre aussi que toute action libre $\Gamma \curvearrowright R$ engendre une action libre à cocycle de Γ sur le facteur II_1 $R' \cap R^\omega$ dont l'annulation en cohomologie est équivalente à la propriété de plongement approximatif de Connes pour $N \rtimes \Gamma$, i.e. $R \rtimes \Gamma R^\omega$.

0. Introduction

A *cocycle action* of a group Γ on a II_1 factor N is a map $\sigma : \Gamma \to \mathrm{Aut}(N)$ which is multiplicative modulo inner automorphisms of N, $\sigma_g\sigma_h = \mathrm{Ad}(v_{g,h})\sigma_{gh}$, $\forall g, h \in \Gamma$, with the unitary elements $v_{g,h} \in \mathcal{U}(N)$ satisfying the cocycle relation $v_{g,h}v_{gh,k} = \sigma_g(v_{h,k})v_{g,hk}$, $\forall g, h, k \in \Gamma$.

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If Γ is a free group $\mathbb{F}_n$, for some $1 \leq n \leq \infty$, then any cocycle Γ-action on any II_1 factor N can obviously be perturbed by inner automorphisms $\{\mathrm{Ad}(w_g)\}_g$ of N so that to become a "genuine" action, i.e., such that $\sigma'_g = \mathrm{Ad}(w_g)\sigma_g$ is a group morphism, in fact so that the stronger condition $v_{g,h} = \sigma_g(w_h^*)w_g^*w_{gh}$, $\forall g, h$, holds true. We obtain in this paper several results towards identifying the class $\mathcal{VC}$ of all countable groups Γ that satisfy this universal vanishing cohomology property. Thus, we first prove that any free product of amenable groups amalgamated over a common finite subgroup is in the class $\mathcal{VC}$. Then we show that if a group Γ has an infinite subgroup which either has relative property (T), or has non-amenable centralizer, then Γ is not in $\mathcal{VC}$. To prove that all amenable groups lie in $\mathcal{VC}$ we use subfactor techniques to reduce the problem to the case N is the hyperfinite II_1 factor R, where vanishing cohomology holds due to results in ([27]). To exclude groups from being in $\mathcal{VC}$ we apply W*-rigidity results to two types of cocycle actions that are "hard to untwist": the ones arising from t-amplifications of Bernoulli actions on $N = R$ introduced in [43]; and the ones considered in [8], arising from normal inclusions $\mathbb{F}_\infty \hookrightarrow \mathbb{F}_n$ with $\mathbb{F}_n/\mathbb{F}_\infty = \Gamma$, which give cocycle Γ-actions on $N = L(\mathbb{F}_\infty)$.

Untwisting cocycle actions on II_1 factors is a basic question in non-commutative ergodic theory and very specific to this area. Besides its intrinsic interest, the problem occurs in the classification of group actions on II_1 factors ([5, 18, 27, 43]) and, closely related to it, in the classification of factors through unique crossed-product decomposition (as in [5] for amenable factors, or [43, 44, 46, 17, 49] for non-amenable II_1 factors). Another aspect, which goes back to ([8]) and is important in W*-rigidity, relates non-vanishing cohomology for certain cocycle Γ-actions on $L(\mathbb{F}_\infty)$ to non-embeddability of $L(\Gamma)$ into $L(\mathbb{F}_n)$. From an opposite angle, which offers a new point of view much emphasized here, vanishing cohomology results for cocycle actions are relevant to embedding problems, such as finding unusual group factors that embed into $L(\mathbb{F}_2)$ and Connes Approximate Embedding conjecture.

To describe the results in this paper in more details we need some background and notations. Let us first note that cocycle actions are more restrictive than *outer actions*, which are maps $\sigma : \Gamma \to \mathrm{Aut}(N)$ that only require $\sigma_g\sigma_h\sigma_{gh}^{-1} \in \mathrm{Inn}(N)$, $\forall g, h \in \Gamma$. It has in fact been shown in ([25]) that there is a scalar 3-cocycle $\nu_\sigma \in H^3(\Gamma)$ associated to an outer action σ. If σ is *free*, i.e., $\sigma_g \notin \mathrm{Inn}(N)$, $\forall g \neq e$, then ν_σ is trivial if and only if σ is a cocycle action. Thus, if we view the vanishing cohomology problem as a question about lifting a 1 to 1 group morphism $\sigma : \Gamma \to \mathrm{Out}(N)$ to a group morphism into $\mathrm{Aut}(N)$, then the problem is not well posed unless one requires $\nu_\sigma \equiv 1$, i.e., that σ defines a cocycle action.

Like for genuine actions, one can associate to a cocycle action $\Gamma \curvearrowright^\sigma N$ a tracial crossed product von Neumann algebra $N \rtimes \Gamma$, with the freeness of σ equivalent to the condition $N' \cap N \rtimes \Gamma = \mathbb{C}1$. Thus, if σ is free then $N \subset M = N \rtimes \Gamma$ is an irreducible inclusion of II_1 factors with the normalizer of N in M generating M as a von Neumann algebra (N is *regular* in M). Conversely, any irreducible regular inclusion of II_1 factors $N \subset M$ arises this way, from a crossed product construction involving a free cocycle action (cf. [18]).

The crossed product framework allows an alternative formulation of vanishing cohomology. Thus, if $M = N \rtimes \Gamma$ denotes the crossed product II_1 factor associated with the free cocycle action (σ, v) of Γ on N, and we let $\{U_g\}_g \subset M$ denote the canonical unitaries implementing σ on N, then the existence of $w_g \in \mathcal{U}(N)$ such that $v_{g,h} = w_g\sigma_g(w_h)w_{gh}^*$, $\forall g, h$

(i.e., *vanishing cohomology* for v) amounts to $U'_g = w_g U_g$ being a Γ-representation. While the condition that $\sigma'_g = \mathrm{Ad}(U'_g)$, $g \in \Gamma$, is a genuine action (i.e., *weak vanishing cohomology* for v) amounts to the weaker condition that $\{U'_g\}_g$ is a projective Γ-representation.

Given a II_1 factor N, we denote by $\mathcal{VC}(N)$ (respectively $\mathcal{VC}_w(N)$) the class of countable groups Γ with the property that any free cocycle action of Γ on N satisfies the strong form (respectively weak form) of the vanishing cohomology. Also, we denote by $\mathcal{VC}$ (respectively $\mathcal{VC}_w$) the class of countable groups Γ with the property that any free cocycle action of Γ on any II_1 factor N satisfies the strong form (respectively weak form) of the vanishing cohomology.

The class $\mathcal{VC}$ contains all finite groups by ([18, 52]) and all groups with polynomial growth by ([36]). The first main result in this paper, which we prove in Section 2, shows that in fact $\mathcal{VC}$ contains all countable amenable groups. Since by [18] all 1-cocycles for actions of finite groups are co-boundaries, this allows to deduce that, more than just containing the free groups, all amalgamated free products of amenable groups over finite groups belong to $\mathcal{VC}$.

THEOREM 0.1. – *The class $\mathcal{VC}$ contains all countable amenable groups. Also, if $\{\Gamma_n\}_n$ is a sequence of groups in $\mathcal{VC}$ and $K \subset \Gamma_n$ is a common finite subgroup, $n \geq 1$, then $\Gamma_1 *_K \Gamma_2 *_K \cdots \in \mathcal{VC}$.*

To prove the first part of this result we show that any cocycle action σ of a countable amenable group Γ on a separable II_1 factor N can be perturbed by inner automorphisms to a cocycle action σ' that leaves invariant an irreducible hyperfinite subfactor $R \subset N$ with the additional property that $\sigma'_g \sigma'_h {\sigma'_{gh}}^{-1}$ are implemented by unitaries in R, $\forall g, h$, with σ' still free when restricted to R (see Theorem 2.1). This reduces the vanishing cohomology problem to the case $N = R$, where one can apply the vanishing cohomology result in ([27]) to finish the proof.

To prove the existence of a "large" $R \subset N$ that's normalized by inner perturbations of σ we use an idea introduced in ([36]; cf. also 5.1.5 in [38]), of translating the problem into the question of whether there exists a sub-inclusion of hyperfinite factors inside the "diagonal subfactor" $N \simeq N^\sigma \subset M^\sigma$ associated with σ, so that to have a non-degenerate commuting square satisfying a *strong smoothness* condition on higher relative commutants. This subfactor problem was solved in [36] in the case Γ is finitely generated with trivial Poisson boundary (e.g., with polynomial growth; see [23]), by constructing R as a limit of relative commutants $P'_n \cap N$ of factors in a tunnel $N \supset N_1 \supset \cdots$, obtained by iterating the downward basic construction (in the spirit of [38, 37]).

However, that construction depends crucially on the trivial Poisson boundary condition on Γ. We use here the amenability of Γ alone to construct a more elaborate decreasing sequence of subfactors $P_n \subset N$ with $P'_n \cap N \nearrow R$ "large" in N, obtained through *reduction/induction* in Jones tunnels. In fact, this method allows us to obtain the existence of strongly smooth embedding of hyperfinite subfactors into any finite index subfactor $N \subset M$ with standard invariant $\mathcal{G}_{N \subset M}$ *amenable* (in the sense of [38], i.e., with its graph $\Gamma_{N \subset M}$ satisfying the Kesten-type condition $\|\Gamma_{N \subset M}\|^2 = [M : N]$; see also [37, 40, 41] for other equivalent definitions). We in fact show that given any amenable C*-category $\mathcal{G}$ of endomorphisms of a II_∞ factor $\mathcal{N}$ (viewed here as an outer action of an abstract rigid C*-tensor category), there exists a "large" approximately finite dimensional (AFD) II_∞